Using the notations of $\begin{cases} \text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} \text{ and} \\ \text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} \end{cases}$,

the Green's Theorem can be written as

Vector forms of Green's Theorem

normal form

$$\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \text{div}\,\vec{F}\,dA}$$

or

$$\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \vec{\nabla}\cdot\vec{F}\,dA}$$

tangential form

$$\boxed{\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_D \text{curl}\,\vec{F}\cdot\hat{k}\,dA}$$

or

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_D (\vec{\nabla}\times\vec{F})\cdot\hat{k}\,dA}$$

And Theorem 10 can be written as:

Thm 10' Ω = simply-connected, $\vec{F}\in C^1$. Then

$$\vec{F} \text{ conservative} \iff \text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = 0$$

(check!)

Note that $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$ can be defined on $\mathbb{R}^n$, for any n.

In particular

Def 12' The <u>divergence</u> of $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ is defined to be
$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial L}{\partial z}$$

Then one can easily check the following facts: (Ex!)

For C^2 function f and C^2 vector field $\vec{F}$:

(i) $\vec{\nabla}\times(\vec{\nabla}f) = 0$ (i.e. $\operatorname{curl}\vec{\nabla}f = 0$)

(ii) $\vec{F}$ conservative $\Rightarrow$ $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = 0$

(iii) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$ (i.e. $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$)

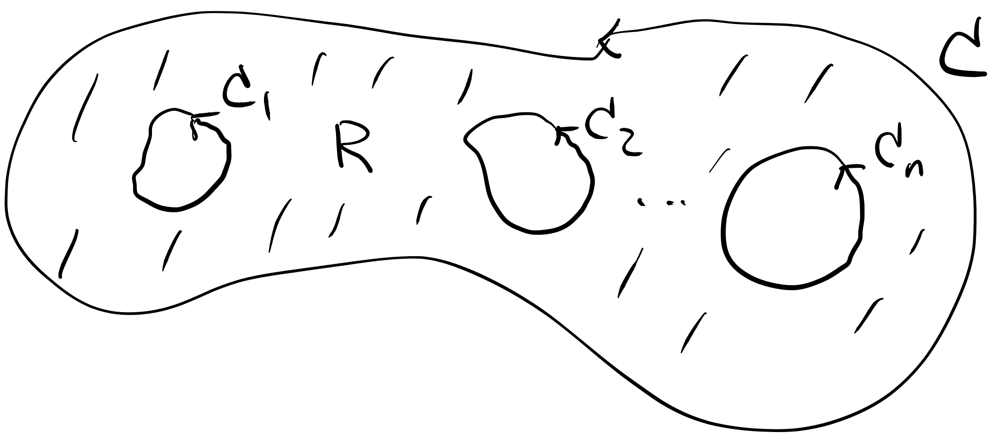
Remark: $\vec{\nabla}\cdot(\vec{\nabla}f) \neq 0$ in general, and it is called the <u>Laplacian</u> of f, and is denoted by
$$\vec{\nabla}^2 f = \vec{\nabla}\cdot(\vec{\nabla}f) = \operatorname{div}(\vec{\nabla}f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

[In graduate level, it will be denoted by $\Delta = \vec{\nabla}^2$ (or $\Delta = -\vec{\nabla}^2$)]

The "operator" $\vec{\nabla}^2$ is called the Laplace operator, and the equation $\vec{\nabla}^2 f = 0$ is called the Laplace equation. Solutions to Laplace equation are called harmonic functions.

In order to apply Green's Theorem to more situations, we need a more general form of Green's Thm:

Suppose we have a simple closed curve C in $\mathbb{R}^2$



Suppose $C_1, C_2, \cdots, C_n$ be pairwise disjoint, piecewise smooth, simple closed curves, such that $C_1, \cdots, C_n$ are enclosed by C.

(All $C, C_1, \cdots, C_n$ are anti-clockwise oriented)

Let R be the region between C and $C_1, \cdots, C_n$.

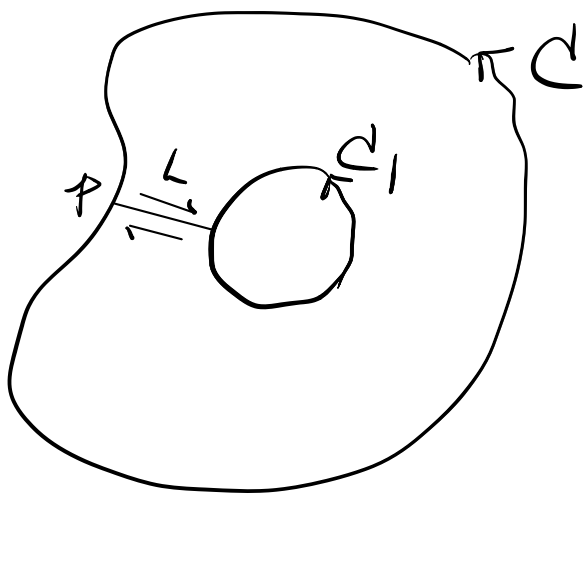
Suppose $\vec{F} = M\hat{i} + N\hat{j}$ is defined on some open set containing R and is C^1. Then, we have

$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy - \sum_{i=1}^{n} \oint_{C_i} M\,dx + N\,dy}$$

(This is the tangential form. The normal form is similar.)

<u>Sketch of Proof:</u>

For simplicity, only one C_1 inside C.



We connect C & C_1 by an "arc" L and consider the "simple" closed curve (starting from p):

$C^* = C + L - C_1 - L$

Then the region R enclosed between C & C_1

is the region enclosed by C^* except the arc L.

Hence $$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA = \iint_{R\setminus L}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA$$

$$\overset{\text{Green's}}{=} \oint_{C^*}(Mdx+Ndy)$$

$$=\left(\oint_C + \oint_L - \oint_{C_1} - \oint_L\right)(Mdx+Ndy)$$

$$= \oint_C (Mdx+Ndy) - \oint_{C_1}(Mdx+Ndy)$$

#

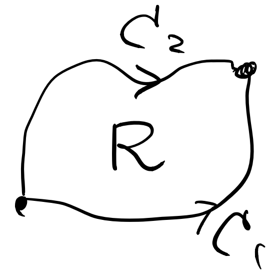
Pf of Thm10 $(n=2)$

We only need to show Ω simple-connected, $\vec{\nabla}\times\vec{F}=0$ $\left(\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\right)$,

then $\vec{F}$ is conservative.

Suppose $C_1, C_2 \subseteq \Omega$ have the same starting point and end point.

Case 1 : C_1 C_2 have no intersection



Then "Ω is simply-connected"

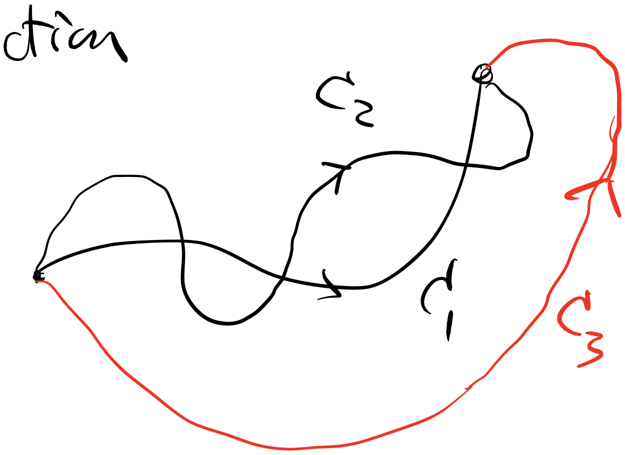
implies the region R enclosed by C_1 and C_2 lies completely in Ω. Then by Green's Thm,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \pm\left(\int_{C_1} - \int_{C_2}\right)(M\,dx + N\,dy)$$

$\underset{0}{\shortparallel}$ by assumption

$$\Rightarrow \int_{C_1} M\,dx + N\,dy = \int_{C_2} M\,dx + N\,dy.$$

<u>Case 2</u> C_1, C_2 have intersection



Pick another curve C_3 with the same starting point and end point and do not intersect C_1 or C_2.

Then by case 1, $\int_{C_1} M\,dx + N\,dy = \int_{C_3} M\,dx + N\,dy$

$$= \int_{C_2} M\,dx + N\,dy$$

$\therefore \int_C \vec{F} \cdot d\vec{r}$ is independent of path,

$\therefore \vec{F}$ is conservative. #

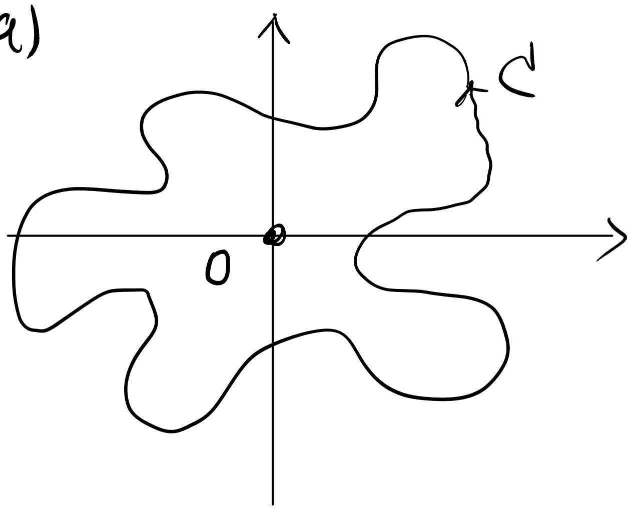
eg 49 $\vec{F} = \underbrace{\frac{-y}{x^2+y^2}}_{M} \hat{i} + \underbrace{\frac{x}{x^2+y^2}}_{N} \hat{j}$ on $\mathbb{R}^2 \setminus \{(0,0)\} = \Omega$.

We've calculated $\oint_{C_1} \vec{F} \cdot d\vec{r} = 2\pi$ for

$$C_1 = x^2+y^2=1$$

How about

(a)



(b)