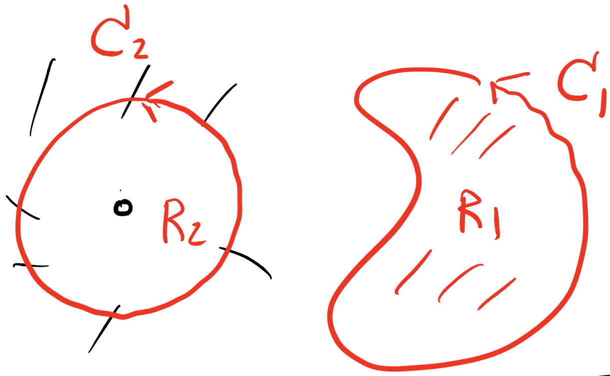
<u>Note</u>: $\underline{\Omega_1} = \mathbb{R}^2 \setminus \{x \le 0\}$

Green's Thm applies as $R \subset \Omega_1$

$\underline{\Omega_2} = \mathbb{R}^2 \setminus \{(0,0)\}$



Let $C_2 = x^2+y^2=1$, then R_2 = region enclosed by C_2

$= \{x^2+y^2<1\} \not\subset \Omega_2$

since $(0,0) \in R_2$ but $(0,0) \notin \Omega_2$

$\therefore$ Green's Thm doesn't apply to C_2 & R_2

$R_1 \subset \Omega_2$ Green's Thm applies

<u>eg 48</u>: Verify both forms of Green's Thm for

$$\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j} \quad \text{on } \Omega = \mathbb{R}^2 \ (\text{is } C^\infty)$$

$$C = \text{unit circle}: \ \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad 0 \le t \le 2\pi$$

Then R = enclosed by $C = \{x^2+y^2<1\}$ the unit disc.

Soln: $M = x-y$, $N = x$ (in this case)

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0.$$

On C, $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$

Normal Form

$$\text{L.H.S.} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} [(\cos t - \sin t)\cos t - \cos t(-\sin t)]\,dt$$

$$= \pi$$

$$\text{R.H.S.} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R 1\,dA = \pi$$

Tangential form

$$\text{L.H.S.} = \oint_C M\,dx + N\,dy$$

$$= \int_0^{2\pi} [(\cos t - \sin t)(-\sin t) + \cos t \cos t]\,dt$$

$$= 2\pi$$

$$\text{R.H.S.} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R [1 - (-1)]\,dA = 2\pi$$

#

Pf of Green's Thm (tangential form)

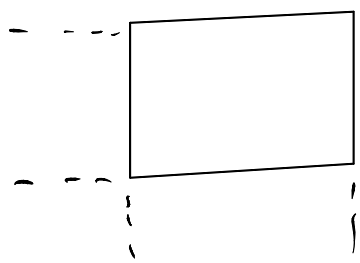
<u>Recall</u>: A region R is of special type:

type (1): If $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$
for some continuous function $g_1(x)$ & $g_2(x)$.

type (2): If $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$
for some continuous function $h_1(x)$ & $h_2(x)$.

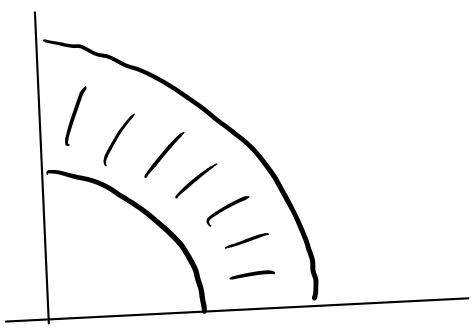
<u>Now</u>: If R is both type (1) and type (2), it said to be <u>simple</u>.

<u>eg 49</u> (i)



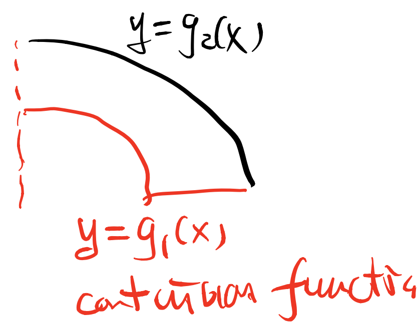
rectangle is simple.

(ii)

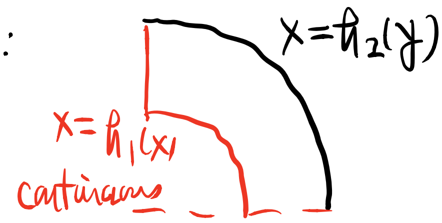


is <u>simple</u> :

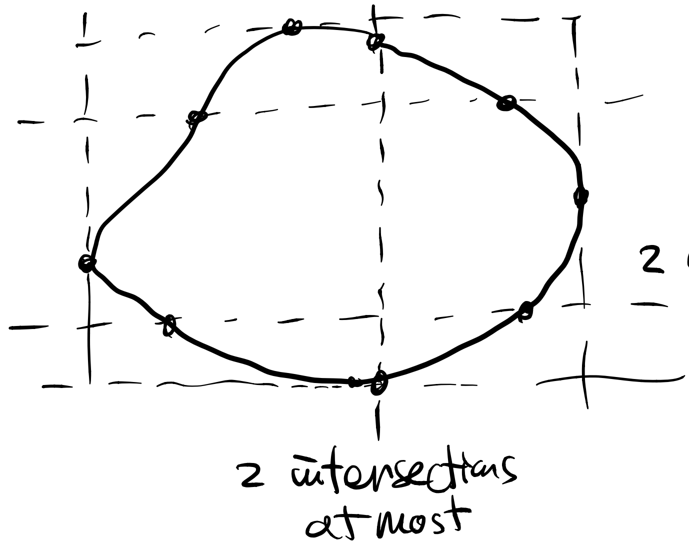
type (1): yes



type (2): yes



(iii)



$$\left.\begin{array}{l}\forall a\in\mathbb{R},\ \#\{\partial R\cap\{x=a\}\}\le 2\\ \text{and } \#\{\partial R\cap\{y=a\}\}\le 2\end{array}\right\}\Rightarrow \text{simple}$$

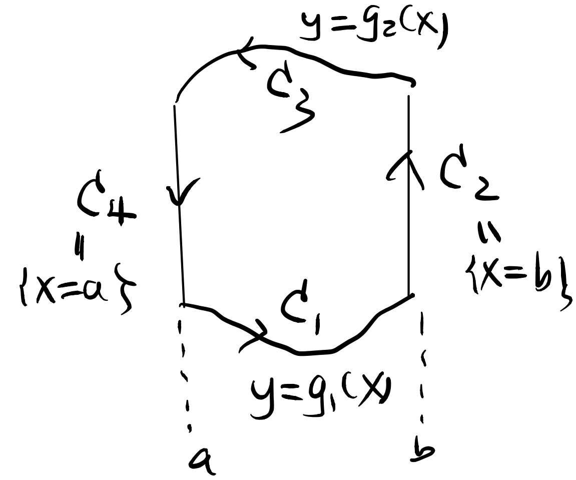
(provided ∂R is piecewise smooth)

Pf of Green Thm for simple region

By definition, R is of type (1) and can be written as

$$R=\{(x,y) : a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$$

Let denote the components of the boundary of R by C_1, C_2, C_3, C_4 as in the figure (note that C_2 and/or C_4 can degenerated to a point.)



Then $\partial R = C_1 + C_2 + C_3 + C_4$ as oriented curve

(using "+" instead of "$\cup$" to denote the orientation)

Now $C_1 = \{ y = g_1(x) \}$ can be parametrized by

$$\vec{r}(t): \ x = t, \ y = g_1(t), \quad a \le t \le b$$

with the correct orientation

$$\therefore \int_{C_1} M\,dx = \int_a^b M(t, g_1(t))\,dt.$$

Similarly "$-C_3$" can be parametrized by

$$\vec{r}(t): \ x = t, \ y = g_2(t), \quad a \le t \le b$$

with the correct orientation

$$\therefore \int_{-C_3} M\,dx = \int_a^b M(t, g_2(t))\,dt$$

$$\Rightarrow \int_{C_3} M\,dx = -\int_a^b M(t, g_2(t))\,dt$$

(oriented)

For $C_2 = \{x = b\}$, it can be parametrized by

$$\vec{r}(t): \ x = b, \ y = t, \quad g_1(b) \le t \le g_2(b)$$

with correct orientation

$\therefore \int_{C_2} M\,dx = 0$

Similarly $\int_{C_4} M\,dx = -\int_{-C_4} M\,dx = 0$

Hence $$\oint_{\partial R} M\,dx = \sum_{i=1}^{4} \int_{C_i} M\,dx$$

$$= \int_a^b [M(t, g_1(t)) - M(t, g_2(t))]\,dt$$

$$\left(= \int_a^b [M(x, g_1(x)) - M(x, g_2(x))]\,dx \quad (x = t) \right)$$

On the other hand, Fubini's Thm $\Rightarrow$

$$\iint_R -\frac{\partial M}{\partial y}\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} -\frac{\partial M}{\partial y}\,dy \right] dx$$

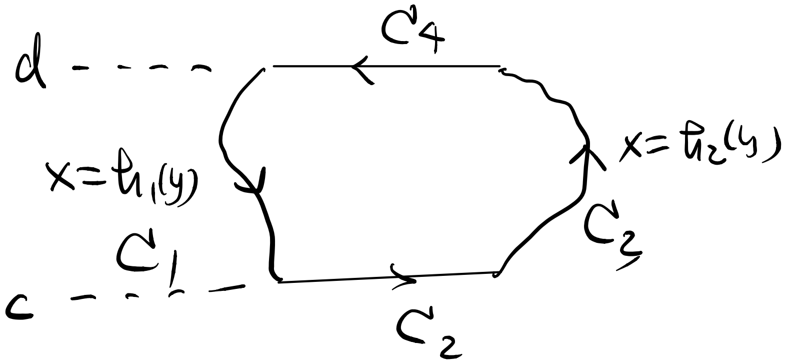
$$= \int_a^b -[M(x, g_2(x)) - M(x, g_1(x))]\,dx$$

$$\therefore \iint_R -\frac{\partial M}{\partial y}\,dA = \oint_{\partial R} M\,dx \,.$$

Since R is also type (2), R can be written as

$$R = \{(x,y) : h_1(y) \leq x \leq h_2(y),\ c \leq y \leq d\}$$

On can show similarly



$$\oint_{\partial R} N\,dy = -\int_c^d N(h_1(t), t)\,dt + 0$$
$$+ \int_c^d N(h_2(t), t)\,dt + 0$$

$$= \int_c^d \left[N(h_2(t), t) - N(h_1(t), t)\right] dt$$

$$= \int_c^d \left[N(h_2(y), y) - N(h_1(y), y)\right] dy \qquad (y=t)$$

$$= \int_c^d \left[\int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}\,dx\right] dy$$

$$= \iint_R \frac{\partial N}{\partial x}\,dA .$$

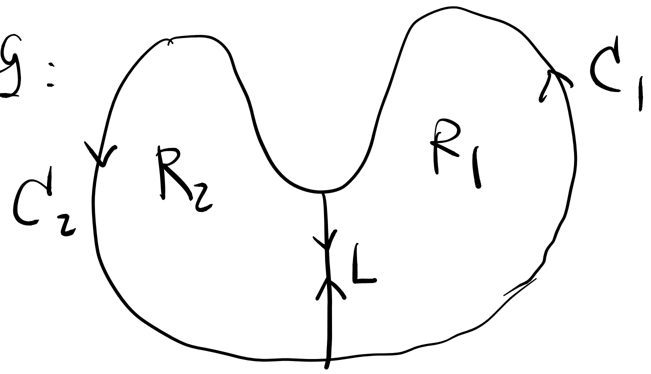
All together $\displaystyle \oint_R (M\,dx + N\,dy) = \iint_R \left(-\frac{\partial M}{\partial y}\right) dA + \iint_R \frac{\partial N}{\partial x}\,dA$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA .$$

Proof of Green's Thm for

$R =$ finite union of simple regions with intersections only along some boundary line segments, and those line segments touch only at the end points at most.

eg:



$R_1, R_2 =$ simple

but $R = R_1 \cup R_2 \neq$ simple

$\partial R_1 = C_1 + L$ ($L: \downarrow$)

$\partial R_2 = C_2 - L$

with anti-clockwise orientation

& $\partial R = C_1 -$

By assumption $R = \bigcup R_i$ finite union s.t.

R_i are simple and

$R_i \cap R_j =$ line segment of a common boundary portion, denote by L_{ij} $(i \neq j)$

Then $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$\left(\begin{array}{l}\text{by Green's thm} \\ \text{for simple region}\end{array}\right) = \sum_i \oint_{\partial R_i} (M dx + N dy)$

Denote C_i = the part of ∂R_i with no intersection with any other R_j (except at end point)

Then $\partial R_i = C_i + \sum_{j \neq i} L_{ij}$

L_{ij} is oriented according to the anti-clockwise orientation of ∂R_i.

Hence

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \int_{C_i + \sum_{j (j \neq i)} L_{ij}} (M dx + N dy)$$

$$= \sum_i \int_{C_i} M dx + N dy$$

$$+ \sum_i \sum_{j (j \neq i)} \int_{L_{ij}} (M dx + N dy)$$

Note that $\sum_i C_i = \partial R$ $\left(\begin{array}{l}\text{as } C_i \text{ is not common boundary} \\ \text{of any other } R_j\end{array}\right)$

$$\therefore \sum_i \int_{C_i} M dx + N dy = \oint_{\partial R} M dx + N dy.$$

Finally, we have

$L_{ji} = -L_{ij}$ as R_i & R_j are located on the two different sides of the common boundary

$$\therefore \sum_i \sum_{\substack{j \\ (j \neq i)}} \int_{L_{ij}} M\,dx + N\,dy = \sum_{\substack{i,j \\ i<j}} \int_{L_{ij}} M\,dx + N\,dy + \sum_{\substack{i,j \\ i>j}} \int_{L_{ij}} M\,dx + N\,dy$$

$$= \sum_{\substack{i,j \\ i<j}} \int_{L_{ij}} M\,dx + N\,dy + \sum_{\substack{j,i \\ j>i}} \int_{L_{ji}} M\,dx + N\,dy$$

$$= \sum_{\substack{i,j \\ i<j}} \left[\left(\int_{L_{ij}} M\,dx + N\,dy \right) + \left(\int_{L_{ji}} M\,dx + N\,dy \right) \right]$$

$$= \sum_{\substack{i,j \\ i<j}} \left(\int_{L_{ij}} M\,dx + N\,dy - \int_{L_{ij}} M\,dx + N\,dy \right)$$

since $L_{ji} = -L_{ij}$

$$= 0 .$$

This 2^{nd} case basically include almost all situations in the level of Advanced Calculus.

The proof of general case needs "analysis" and will be omitted here. #

Def 12: The divergence of $\vec{F} = M\hat{i} + N\hat{j}$ is defined to be

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Note:

$$\operatorname{div} \vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\operatorname{Area}(\overline{D}_\varepsilon^{(x,y)})} \iint_{\overline{D}_\varepsilon(x,y)} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial \bar{D}_\varepsilon(x,y)} \vec{F} \cdot \vec{n}\, ds$$

$$= \text{"flux density"}.$$

<u>Notation</u>: For $f(x,y)$, $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ (gradient)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f$$

It is convenient to denote

$$\boxed{\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}}$$

Then

$$\vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) \cdot \left(M\hat{i} + N\hat{j}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \operatorname{div} \vec{F}.$$

Hence we also write $\boxed{\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}}$

<u>Def 13</u>: Define $\operatorname{rot} \vec{F}$ is be $\operatorname{rot} \vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

$(\vec{F} = M\hat{i} + N\hat{j})$

Note:

$$\text{rot}\,\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon)} \iint_{\bar{D}_\varepsilon} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon)} \oint_{\partial \bar{D}_\varepsilon} \vec{F} \cdot \vec{T}\, ds$$

$$= \text{circulation density}$$

Using $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$, we can write

$$\boxed{\text{rot}\,\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}}$$

since $\vec{F} = M\hat{i} + N\hat{j} + \text{zero}.\hat{k}$ (in $\mathbb{R}^3$)

$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ ("gradient" in $\mathbb{R}^3$)

& $\frac{\partial M}{\partial z} = \frac{\partial N}{\partial z} = 0$

$$\Rightarrow \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$$\Rightarrow \quad \operatorname{rot} \vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}$$

$\hat{k}$-component of "curl $\vec{F}$".

where $\boxed{\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}}$.