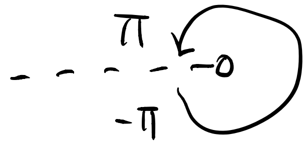
(Cont'd) $\vec{F} = -\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} \frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r} \\ \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r} \end{cases}$$

$$\Rightarrow \vec{F} = \frac{\partial\theta}{\partial x}\hat{i} + \frac{\partial\theta}{\partial y}\hat{j} = \vec{\nabla} f$$

(Recall $f(x,y) = \theta$ which is a smooth function on Ω_1



Ω_1

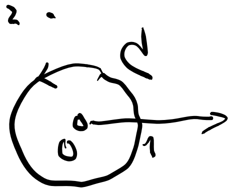
On Ω_2, we have $C : \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $t \in [-\pi, \pi]$
(which is closed curve in Ω_2)

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{-\pi}^{\pi} \left(-\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}\right) \cdot \left(-\sin t\,\hat{i} + \cos t\,\hat{j}\right) dt$$

$$= \int_{-\pi}^{\pi} \left(-\sin t\,\hat{i} + \cos t\,\hat{j}\right) \cdot \left(-\sin t\,\hat{i} + \cos t\,\hat{j}\right) dt$$

$(r = 1$, and $\theta = t$ on C.)

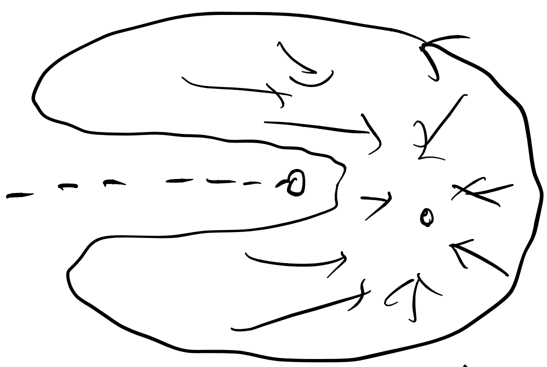
Ω_2:



$$= \int_{-\pi}^{\pi} (\sin^2 t + \cos^2 t)\, dt = \int_{-\pi}^{\pi} dt = 2\pi \neq 0$$

#

Summary:

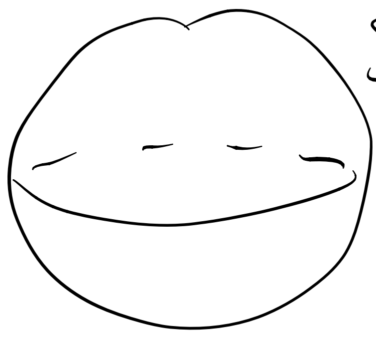
Ω_1	Ω_2
$f(x,y)=\theta$ smooth function on Ω_1	$f(x,y)=\theta$ is not a smooth function on Ω_2 (θ cannot be well-defined)
$C: x^2+y^2=1$ is not even a curve in Ω_1 because $(-1,0)\in C$ $(-1,0)\notin\Omega_1$	$C: x^2+y^2=1$ is a closed curve in Ω_2
 closed curves can not around the origin $\Rightarrow$ closed curves can be defomed continuous (with in Ω_1) to a point.	 C enclosed the "hole" $\Rightarrow$ C cannot be deformed continuous (with in Ω_2) to a point.

Defn 15 A subset $D \subseteq \mathbb{R}^n$, $n = 2$ or 3, is called simply-connected if every closed curve in D can be contracted to a point in D without ever leaving D.

(contracted: deformed continuously)

eg 44: Ω_1 in eg 43 is simply-connected, but Ω_2 is not simply-connected.

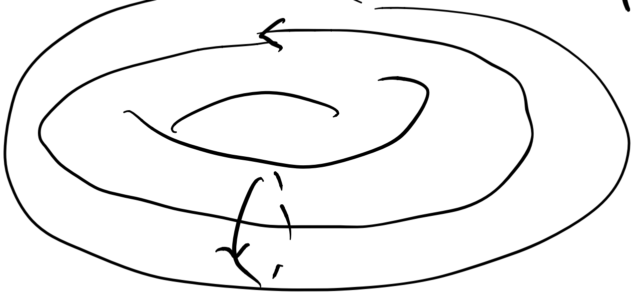
eg 45:



$S^2 \subset \mathbb{R}^3$ $\quad S^2: x^2+y^2+z^2=1$

is simply-connected.

eg 46:



torus $\mathbb{T}^2 \simeq S^1 \times S^1 \subset \mathbb{R}^3$ is not simply-connected.

Remark: Simply connectedness is a global condition to guarantee "eqts in Cor to Thm 8" $\Rightarrow$ "conservative".

Thm 10: Suppose $\Omega \subseteq \mathbb{R}^n$, $n=2$ or 3, is connected and simply-connected. Let $\vec{F}$ be C^1 vector field on Ω. Then

$\vec{F}$ is conservative on Ω

$\Leftrightarrow$ components of $\vec{F}$ satisfy the system of equations in the Cor to the Thm 9.

eg 47 Let $\Omega = \mathbb{R}^3$ (connected and simply connected)

$$\text{Let } \vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$$
$$= (y+e^z)\hat{i} + (x+1)\hat{j} + (1+xe^z)\hat{k}.$$

Find the potential function f of $\vec{F}$, i.e.

$$\vec{\nabla} f = \vec{F}.$$

Soln: This is, we want to find f s.t.

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = L.$$

Checking M, N, L satisfy the system of eqts in Cor to Thm 9:

$$\frac{\partial M}{\partial x} = 0 \qquad \frac{\partial M}{\partial y} = 1 \qquad \frac{\partial M}{\partial z} = e^z$$

$$\frac{\partial N}{\partial x} = 1 \qquad \frac{\partial N}{\partial y} = 0 \qquad \frac{\partial N}{\partial z} = 0$$

$$\frac{\partial L}{\partial x} = e^z \qquad \frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial z} = xe^z$$

Thm 10 $\Rightarrow$ existence of potential function f.

To find f explicitly:

$$\frac{\partial f}{\partial x} = y + e^z$$

$$\Rightarrow \quad f = \int (y+e^z)\,dx = xy + xe^z + \text{"const in } x\text{"}$$

$$= xy + xe^z + g(y,z) \quad \text{for some function in } y, z$$

$$x+1 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy + xe^z + g(y,z)\right)$$

$$= x + \frac{\partial g}{\partial y}$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 1$$

$$\Rightarrow \quad g = y + \text{"const in } y\text{"} = y + h(z)$$

for some function $h(z)$.

$\therefore \quad f = yx + xe^z + y + h(z)$

$$1 + xe^z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(yx + xe^z + y + h(z)\right)$$

$$= xe^z + h'(z)$$

$\Rightarrow \quad h'(z) = 1$

$\Rightarrow \quad h(z) = z + \text{const}$

Hence

$$f(x,y,z) = yx + xe^z + y + z + c$$

where c is a constant.

\#

(Note: This is equivalent to find f s.t.
total differential $df = M\,dx + N\,dy + L\,dz$)

Remark:

To prove Thm 10 in $\mathbb{R}^2$, we need the Green's Thm

(in $\mathbb{R}^3$, we need Stokes' Thm)

<u>Thm 11</u> (Green's Theorem)

Let $\Omega \subseteq \mathbb{R}^2$ be open; $\vec{F} = M\hat{i} + N\hat{j}$ be C^1 vector field on Ω; C is a piecewise <u>smooth</u> <u>simple closed</u> <u>anti-clockwisely oriented</u> curve enclosing a region R which lies entirely in Ω.

Then

- <u>Normal Form</u>

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \boxed{\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}$$

- <u>Tangential Form</u>

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

(<u>Remark</u>: The two forms are equivalent.)