Conservative Vector Field

Def 14 Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be open. A vector field $\vec{F}$ defined on Ω is said to be conservative if

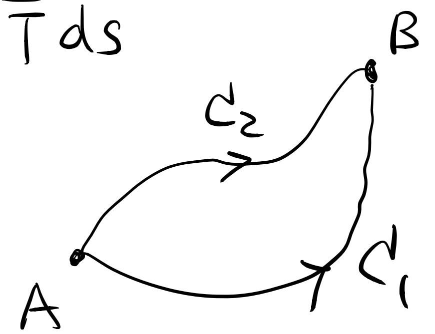
$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r}$$

along an oriented curve C on Ω depends only on the starting point and end point of C.

Note: This is usually referred as "path independent"

i.e. If C_1, C_2 are oriented curves with starting point A and end point B, then

$$\int_{C_1} \vec{F} \cdot \vec{T}\, ds = \int_{C_2} \vec{F} \cdot \vec{T}\, ds$$



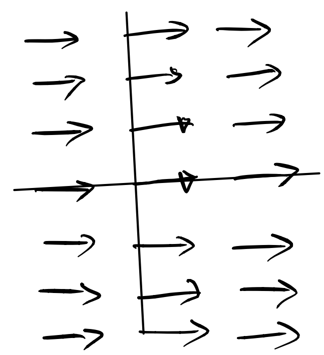
(so the value only depends on the points A and B)

<u>Notation</u> : If $\vec{F}$ is conservative, we sometimes write

$$\int_A^B \vec{F}\cdot\vec{T}\,ds$$ to denote the common value

$\int_C \vec{F}\cdot\vec{T}\,ds$ along any oriented curve C from A to B.

<u>eg41</u> : $\vec{F} \equiv \hat{i}$ on $\mathbb{R}^2$



C : $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$a \le t \le b$

Then $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

$$= \int_a^b x'(t)\,dt = x(b) - x(a)$$

x-coordinates of the starting point and end point of C.

$\Rightarrow$ $\int_C \vec{F}\cdot\vec{T}\,ds$ depends only on the starting & end points

$\Rightarrow$ $\vec{F}$ is conservative.

(Note $\vec{F} = \vec{\nabla} f$, where $f(x,y) = x$)

Thm 8 (Fundamental Theorem of Line Integral)

Let f be a C^1 function on an open set $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, and $\vec{F} = \vec{\nabla} f$ be gradient vector field of f.

Then for any piecewise smooth oriented curve C on Ω with starting point A and end point B,

$$\int_C \vec{F} \cdot \vec{T} \, ds = f(B) - f(A)$$

($\therefore$ gradient $\Rightarrow$ conservative)

Pf: First assume C is smooth with parametrization $\vec{r}(t)$, $a \le t \le b$.

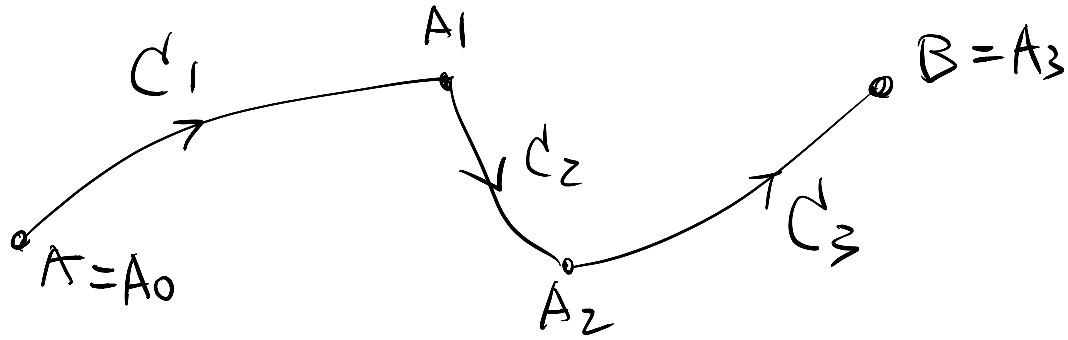
Then
$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{F} \cdot d\vec{r}$$
$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$
$$= \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$
chain rule
$$= \int_a^b \frac{d}{dt} f(\vec{r}(t)) \, dt$$
$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(B) - f(A)$$

For a general piecewise smooth curve

$$C = C_1 \cup C_2 \cup \cdots \cup C_k$$

$\Big($ $= C_1 + C_2 + \cdots + C_k$ in order to indicate the orientation of C_i are correct wrt the orientation of C $\Big)$

(better notation)



where C_i is smooth going from A_{i-1} to A_i.

Then
$$\int_C \vec{F}\cdot\vec{T}\,ds = \sum_i \int_{C_i} \vec{F}\cdot\vec{T}\,ds$$
$$= \sum_i \left[f(A_i) - f(A_{i-1})\right]$$
$$= f(A_k) - f(A_0)$$
$$= f(B) - f(A) \quad \#$$

Thm 9 Let $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, be open and connected.

$\vec{F}$ is a continuous vector field on Ω. Then the followings are equivalent.

(a) $\exists$ a C^1 function $f: \Omega \to \mathbb{R}$ such that

$$\vec{F} = \vec{\nabla} f$$

(b) $\oint_C \vec{F} \cdot d\vec{r} = 0$ along any closed curve C on Ω.

(c) $\vec{F}$ is conservative.

Pf "a $\Rightarrow$ b" If f is C^1 and $\vec{F} = \vec{\nabla} f$,

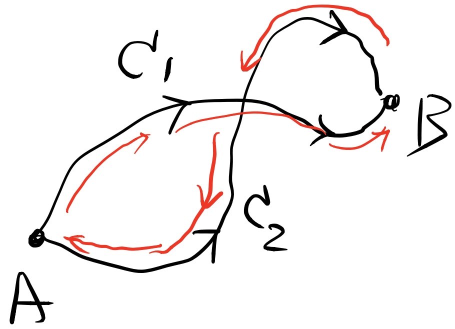
and $\vec{r} = [a,b] \to \Omega$ parameterizes C.

C closed $\Rightarrow$ $\vec{r}(a) = \vec{r}(b) = A$

Fundamental Thm of Line Integral

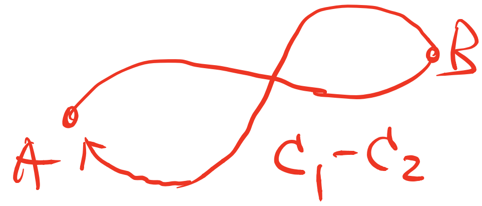
$$\Rightarrow \oint_C \vec{F} \cdot \vec{T} ds = f(\vec{r}(b)) - f(\vec{r}(a))$$
$$= f(A) - f(A) = 0.$$

"b ⇒ c" Suppose C_1, C_2 are oriented curves with starting point A and end points B.



Then $C_1 \cup (-C_2)$

(a letter denoted by $C_1 - C_2$)



(or $C_1 \cup (-C_2)$)

Then $C_1 - C_2$ is an oriented closed curve. Hence by (b), we have

$$0 = \oint_{C_1 - C_2} \vec{F} \cdot \vec{T} \, ds = \int_{C_1} \vec{F} \cdot \vec{T} \, ds + \int_{-C_2} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_{C_1} \vec{F} \cdot \vec{T} \, ds - \int_{C_2} \vec{F} \cdot \vec{T} \, ds$$

$$\therefore \quad \int_{C_1} \vec{F} \cdot \vec{T} \, ds = \int_{C_2} \vec{F} \cdot \vec{T} \, ds$$

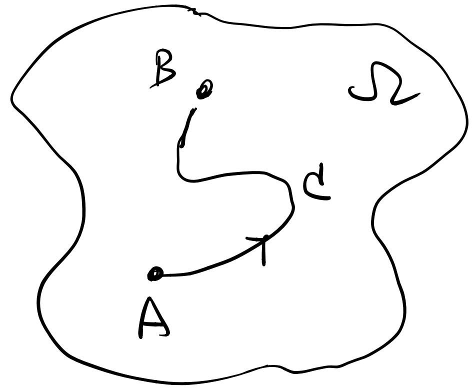
Since C_1, C_2 are arbitrary, $\vec{F}$ is conservative.

"(c) ⇒ (a)" Assume $n = 2$ for simplicity (other dimensions are similar)

Let $\vec{F} = M\hat{i} + N\hat{j}$ be conservative.

Fix a point $A \in \Omega$.

For any point $B \in \Omega$,



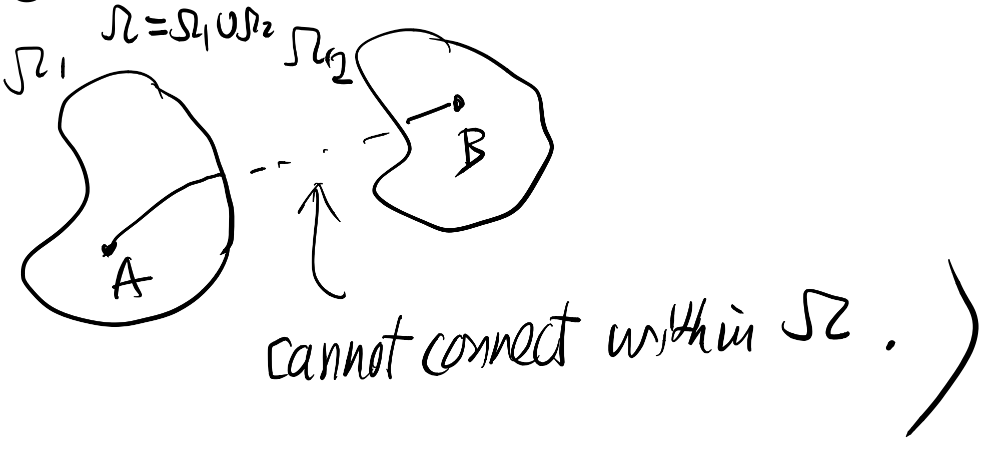
define

$$f(B) = \int_A^B \vec{F} \cdot \vec{T} \, ds$$

$$= \int_C \vec{F} \cdot d\vec{r}$$ for C is an oriented curve from A to B

Since $\vec{F}$ is conservative $\Rightarrow$ $\int_A^B \vec{F} \cdot \vec{T} \, ds$ is independent of C

(We've also used the assumption that Ω is connected, otherwise there is no path from A to B if A, B belong to different connected components:



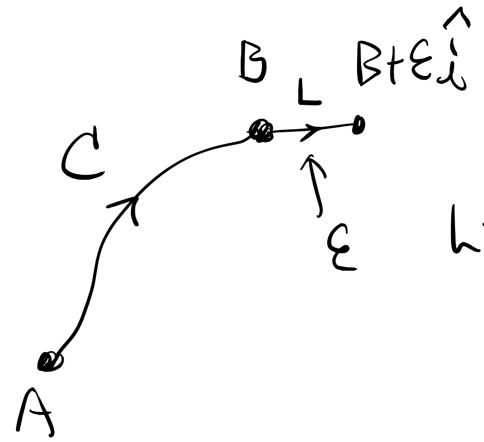
)

Hence $f(B)$ is well-defined.

Claim $\vec{F} = \vec{\nabla} f$.

Pf of claim: $\frac{\partial f}{\partial x}(B) = \lim_{\varepsilon \to 0} \frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon}$

Let C be an oriented curve from A to B



L = horizontal.

Then

$f(B+\varepsilon\hat{i})$

$$= \int_A^{B+\varepsilon\hat{i}} \vec{F} \cdot d\vec{r}$$

$$= \int_{C+L} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_L \vec{F} \cdot d\vec{r}$$

$$= \int_A^B \vec{F} \cdot d\vec{r} + \int_L \vec{F} \cdot d\vec{r}$$

$$= f(B) + \int_L \vec{F} \cdot d\vec{r}$$

$$\Rightarrow f(B+\varepsilon\hat{i}) - f(B) = \int_L \vec{F} \cdot d\vec{r}$$

$$= \int_0^{\varepsilon} M(x+t, y)\, dt$$

where $B = (x, y)$

(using parametrization

$$\vec{r}(t) = (x+t)\hat{i} + y\hat{j} \quad , \quad 0 \le t \le \varepsilon$$

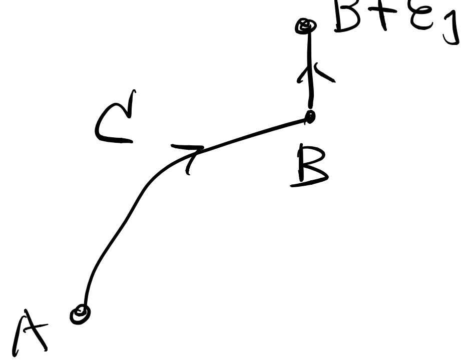
$$\Rightarrow \quad \vec{r}\,'(t) = \hat{i} \quad)$$

$$\Rightarrow \lim_{\varepsilon \to 0} \frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} M(x+t, y)\, dt$$

$$= M(x,y)$$

$$\therefore \quad \frac{\partial f}{\partial x}(x,y) = M(x,y).$$

Similarly $\frac{\partial f}{\partial y}(x,y) = N(x,y)$ by considering the curve :



So $\vec{\nabla} f = \vec{F}$.

Note that $\vec{F}$ is continuous, i.e. $M(x,y)$ & $N(x,y)$ are continuous $\Rightarrow$ $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous

$$\Rightarrow f \text{ is } C^1 \quad \#$$

Remark: The function f in (a) of Thm 9 is called the potential function for $\vec{F}$. It is unique up an additive constant:

$$\vec{\nabla}(f+c)=\vec{F}, \quad \forall\ c=\text{const.}$$

Corollary (to Thm 9)

Let $\vec{F}$ be conservative and C^1

($n=3$) If $\vec{F}=M\hat{i}+N\hat{j}+L\hat{k}$ (on $\Omega\subset\mathbb{R}^3$)

then
$$\begin{cases} \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x} \\[2ex] \dfrac{\partial N}{\partial z}=\dfrac{\partial L}{\partial y} \\[2ex] \dfrac{\partial L}{\partial x}=\dfrac{\partial M}{\partial z} \end{cases}$$

($n=2$) If $\vec{F}=M\hat{i}+N\hat{j}$, then (on $\Omega\subset\mathbb{R}^2$)

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Pf: $\vec{F}$ conservative $\xrightarrow{\text{Thm 9}}$ $\vec{F} = \vec{\nabla} f$ for some function f

i.e. $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= M\hat{i} + N\hat{j} + L\hat{k} = \vec{F}$

$\Leftrightarrow \quad M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}, \quad L = \frac{\partial f}{\partial z}$

Mixed derivatives Thm (Clairaut's Thm)

$$\vec{F} \in C^1 \Rightarrow \begin{cases} \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial L}{\partial y} \\ \frac{\partial L}{\partial x} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial M}{\partial z} \end{cases}$$

\#

eg 42: Show that $\vec{F}(x,y) = \hat{i} + x\hat{j}$ is not conservative on $\mathbb{R}^2$

Soln: $M \equiv 1,\ N = x \Rightarrow \frac{\partial M}{\partial y} = 0 \neq 1 = \frac{\partial N}{\partial x}$

$\Rightarrow \vec{F}$ is not conservative. \#

Remark : (Important)

For a C^1 vector field $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$\boxed{\vec{F} \text{ conservative}}$ $\overset{\text{Cor to Thm 9}}{\Longrightarrow}$ $\boxed{M, N, L \text{ satisfy the system eqts in Cor to Thm 9.}}$

$\underset{?}{\Longleftarrow}$

Answer: Not true in general, needs extra condition on the domain Ω.

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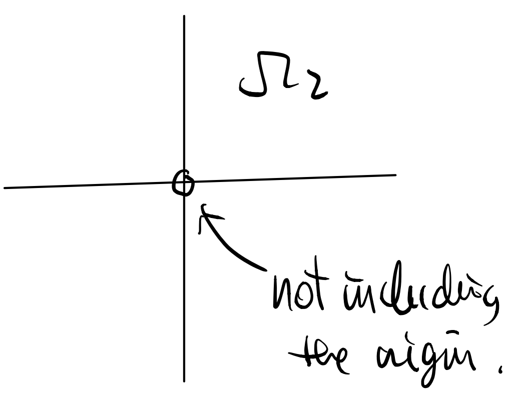
eg43 Consider the vector field

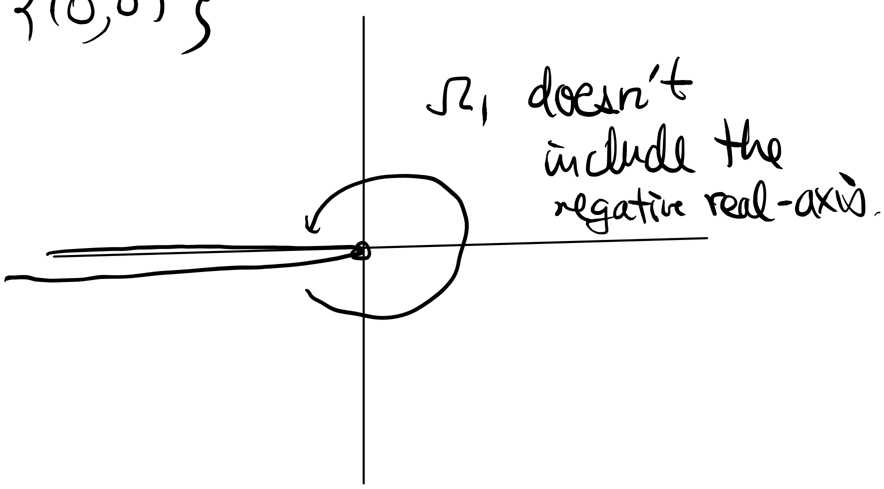
$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$$

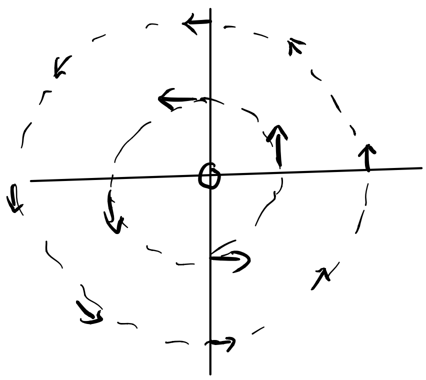
and the domains

$$\Omega_1 = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \leq 0\}$$

$$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$$







In polar coordinate

$$\vec{F} = \frac{-\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}$$

$$\begin{cases} \vec{F} \text{ rotates around the origin anti-clockwisely.} \\ |\vec{F}| = \frac{1}{r} \to 0 \text{ as } r \to \infty, \end{cases}$$

- $|\vec{F}| \uparrow +\infty$ as $r \to 0$, so $\vec{F}$ cannot be extended to a C^1 vector field on $\mathbb{R}^2$.

Questions: Is $\vec{F}$ conservative on Ω_1?

Is $\vec{F}$ conservative on Ω_2?

Soln: (1) For Ω_1, any (x,y) can be expressed in polar coordinate with

$$\begin{cases} r > 0 \\ -\pi < \theta < \pi \end{cases} \quad ((r,\theta) \text{ are unique})$$

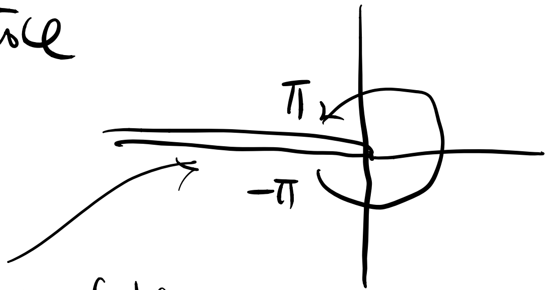
Define $f(x,y) = \theta$ smooth on Ω_1

Then $\vec{\nabla} f = \vec{F}$ (Check!)

(Thm 9)
$\Rightarrow$ $\vec{F}$ is conservative on Ω_1.

(2) For Ω_2, the function $f(x,y)=\theta$ cannot be extended to a smooth function on Ω_2,

since



jump of the value of f $\Rightarrow$ f cannot be extended to a "continuous" on Ω_2

$\therefore$ $f(x,y)=\theta$ doesn't work in the case of Ω_2

We can check, for C : $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$
(unit circle $t \in [-\pi, \pi]$)

then

$$\oint_C \vec{F} \cdot d\vec{r} = 2\pi \quad \text{(check)}$$
$$\neq 0$$

$\therefore$ Thm 9 $\Rightarrow$ $\vec{F}$ is not conservative on Ω_2.