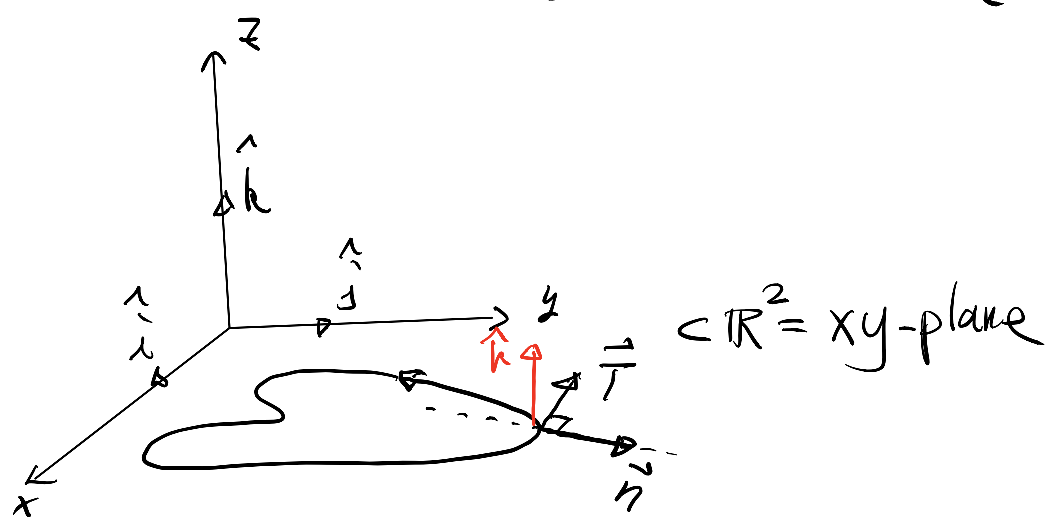
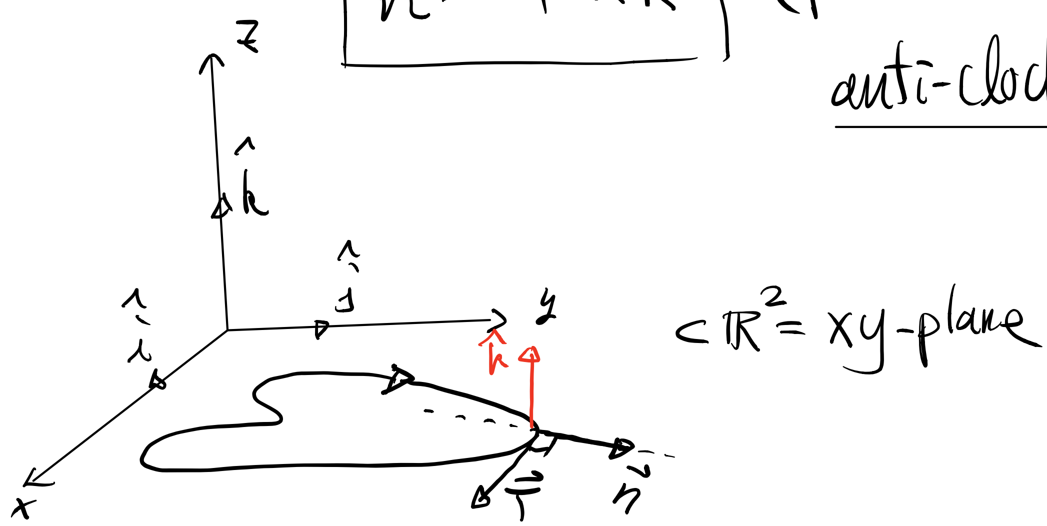
(3) $\vec{F}$ = velocity of fluid

C = oriented plane curve ($C \subset \mathbb{R}^2$)
parametrized by
$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$\vec{n}$ (or $\hat{n}$) = outward-pointing unit normal vector to the curve C



$$\boxed{\vec{n} = \vec{T} \times \hat{k}}$$ (provided C is of anti-clockwise orientation)



$$\boxed{\vec{n} = -\vec{T} \times \hat{k}}$$ (provided C is of clockwise orientation)

Formula for $\vec{n}$ (wrt the parametrization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$)

Recall
$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{x'(t)\hat{i} + y'(t)\hat{j}}{|\vec{r}'(t)|}$$
$$\left(= \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} \right)$$
(where s = arc-length parameter and $ds = |\vec{r}'(t)|\,dt$)

Anti-clockwise :
$$\vec{n} = \vec{T} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$$
$$= \frac{y'(t)\hat{i} - x'(t)\hat{j}}{|\vec{r}'(t)|}$$

Clockwise :
$$\vec{n} = -\vec{T} \times \hat{k} = -\frac{dy}{ds}\hat{i} + \frac{dx}{ds}\hat{j}.$$

<u>Flux</u> of $\vec{F}$ across $C \overset{\text{def}}{=} \int_C \vec{F}\cdot\vec{n}\, ds$ (Amount of fluid getting out of the closed curve C)

If $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$

and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is <u>anti-clockwise</u> parametrization of C.

Then

$$\boxed{\begin{aligned} &\text{Flux of } \vec{F} \text{ across } C \\ &= \oint_C (M\hat{i}+N\hat{j})\cdot\left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds \\ &= \oint_C M\,dy - N\,dx \end{aligned}}$$

" $\oint$ " means the curve is <u>closed</u> and in <u>anti-clockwise</u> orientation.

Similarly, " $\oint$ " means the curve is <u>closed</u> and in <u>clockwise</u> orientation.

Remark: We usually refer the anti-clockwise orientation as the positive orientation of a closed curve in $\mathbb{R}^2$ (wrt the "orientation" of $\mathbb{R}^2$)

eg 40 Let $\vec{F} = (x-y)\hat{i} + x\hat{j}$

$C: x^2+y^2=1$

Find the flow (anti-clockwisely) along C and flux across C.

Soln: Let $\vec{r}(t) = \underset{\left(\overset{\shortparallel}{x(t)}\right.}{\cos t}\,\hat{i} + \underset{\left.\overset{\shortparallel}{y(t)}\right)}{\sin t}\,\hat{j}$, $0 \le t \le 2\pi$.

Then flow $= \oint_C \vec{F}\cdot\vec{T}\,ds$ $\quad$ (Recall $\vec{T}ds = d\vec{r}$)

$= \oint_C \vec{F}\cdot d\vec{r}$

$= \int_0^{2\pi} \left[(\cos t - \sin t)\hat{i} + \cos t\,\hat{j}\right]\cdot\left[-\sin t\,\hat{i} + \cos t\,\hat{j}\right]dt$

$= \int_0^{2\pi} (-\cos t\sin t + 1)\,dt = 2\pi$ (check!)

$$\underline{\text{Flux}} = \oint_C \vec{F} \cdot \hat{n}\, ds$$

$x = \cos t$

$y = \sin t$

$M = \cos t - \sin t$

$N = \cos t$

$$= \oint_C M\,dy - N\,dx$$

$$= \oint_C \left[(\cos t - \sin t)\cos t - \cos t(-\sin t)\right] dt$$

$$= \int_0^{2\pi} \cos^2 t\, dt = \pi$$

$\underline{\text{Remark}}$: If C is an <u>oriented curve</u>, denote by $-C$ the <u>oriented</u> curve with <u>opposite</u> orientation.

If f is a <u>scalar</u> function

$$\boxed{\int_C f\,ds = \int_{-C} f\,ds}$$

as "ds" is not oriented, just means "length"

If $\vec{F}$ is a vector field

$$\boxed{\int_C \vec{F} \cdot \vec{T}_C\, ds = -\int_{-C} \vec{F} \cdot \vec{T}_{-C}\, ds}$$

Hence $\vec{T}_C$ = unit tangent vector of C

$\vec{T}_{-C}$ = unit tangent vector of $-C$.

& $\boxed{\vec{T}_C = -\vec{T}_{-C}}$

But for flux

$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_{-C} \vec{F}\cdot\vec{n}\,ds$$

same outward pointing unit normal vector.

$\therefore$ $\boxed{\text{Flux doesn't depend on the orientation of } C.}$