Remark for eg 37:

If we use $ds = |\vec{r}'(t)|\,dt$, then

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds}$$ derivative wrt "arc-length",

where "arc-length s" is definited (up a additive constant) by

$$s(t) = \int_{t_0}^{t} |\vec{r}'(t)|\,dt .$$

A parametrization of a curve C by arc length s is called arclength parametrization.

($\vec{r}(s)$ = arc length parametrization, then $\left|\frac{d\vec{r}}{ds}(s)\right| \equiv 1$.)

Ref 11 A vector field is defined to be continuous/differentiable/C^k if the component functions are.

eg 38: $\vec{F}(x,y) = x\hat{i} + y\hat{j}$ is C^∞, but

$\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ is not continuous in $\mathbb{R}^2$.

Line integral of vecta field

Def 12: Let C be a curve with orientation given by a parametrization $\vec{r}(t)$ with $\vec{r}'(t) \neq \vec{0}$, $\forall t$. Define the line integral of a vecta field $\vec{F}$ along C to be

$$\int_C \vec{F} \cdot \vec{T} \, ds,$$

where $\vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ is the unit tangent vecta field along C.

Note: If $\vec{r} : [a,b] \to \mathbb{R}^n$, ($n = 2$ or 3) then

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_a^b \underbrace{\vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|}}_{\vec{F} \cdot \vec{T}} \underbrace{|\vec{r}'(t)| dt}_{ds}$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

Hence we also write $\boxed{\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{F} \cdot d\vec{r}}$

eg38: $\vec{F}(x,y,z) = z\hat{i} + xy\hat{j} - y^2\hat{k}$

$C: \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}, \quad 0 \le t \le 1.$

$\left(= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \right)$

Then $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

$$= \int_0^1 \left(\sqrt{t}\,\hat{i} + (t^2\cdot t)\hat{j} - t^2\hat{k}\right)\cdot \left(2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right) dt$$

$$= \int_0^1 \left(2t\sqrt{t} + t^3 - \frac{t^{3/2}}{2}\right) dt = \frac{17}{20}$$

Line Integral of $\vec{F} = M\hat{i} + N\hat{j}$ along

$C: \vec{r}(t) = g(t)\hat{i} + h(t)\hat{j}$ can be expressed as

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$$

$$= \int_a^b \left[M g'(t) + N h'(t)\right] dt$$

Similarly for $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ along C:

$\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + f(t)\vec{k}$ is

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b [Mg'(t) + Nh'(t) + Lf'(t)]\,dt.$$

<u>Note</u>: Usually, people write
$$dx = g'(t)\,dt$$
$$dy = h'(t)\,dt$$
$$dz = f'(t)\,dt$$

$\therefore$ The line integral can be denoted by

$$\boxed{\int_C \vec{F}\cdot\vec{T}\,ds = \int_C M\,dx + N\,dy + L\,dz}$$

(Similarly for $\mathbb{R}^2$ situation)

One can also think of $\vec{r} = (x, y, z)$ is the position vector (field) and

$$\boxed{d\vec{r} = (dx, dy, dz)}$$

then
$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C (M, N, L)\cdot(dx, dy, dz)$$
$$= \int_C M\,dx + N\,dy + L\,dz.$$

eg 39 : Evaluate $I = \int_C -y\,dx + z\,dy + 2x\,dz$

where $C : \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k} \quad (0 \leq t \leq 2\pi)$

$$\left(= (\cos t, \sin t, t) \right)$$

Soln: $x = \cos t$, $y = \sin t$, $z = t$

$$d\vec{r} = (-\sin t, \cos t, 1)\,dt$$

$$\Rightarrow \quad I = \int_0^{2\pi} [\sin t\,(-\sin t) + t\cos t + 2\cos t]\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + t\cos t + 2\cos t)\,dt$$

$$= \pi .$$

Physics

(1) $\vec{F}$ = Force field

C = oriented curve,

then $\boxed{W = \int_C \vec{F} \cdot \vec{T}\,ds}$

is work done in moving an object along C.

(2) $\vec{F}$ = velocity vector field of <u>fluid</u>

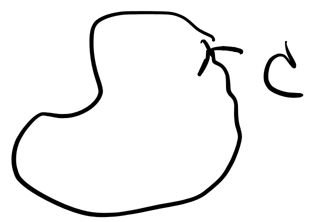
C = oriented curve.

Then

$$\boxed{\text{Flow} = \int_C \vec{F}\cdot\vec{T}\,ds}$$

<u>Flow</u> along the curve C.

If C is a <u>closed</u> curve, the <u>flow</u> is also called the <u>circulation</u>.

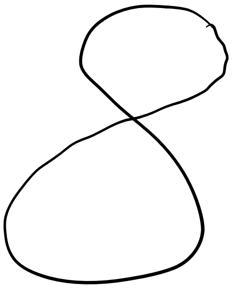
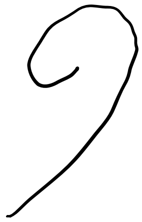
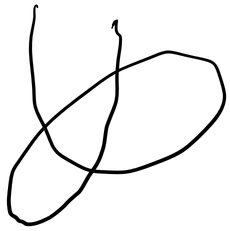
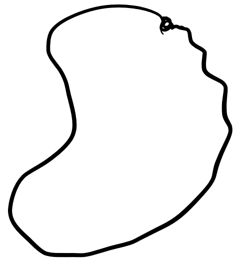


<u>Def 13</u> A curve is said to be

(i) <u>simple</u> if it <u>does not</u> intersect with itself except possibly at end points.

(ii) <u>closed</u> if starting point = end point. (also called a <u>loop</u>)

(iii) <u>simple closed curve</u> if it is both simple and closed.

Note:				
simple	No	Yes	No	Yes
closed	Yes	No	No	Yes