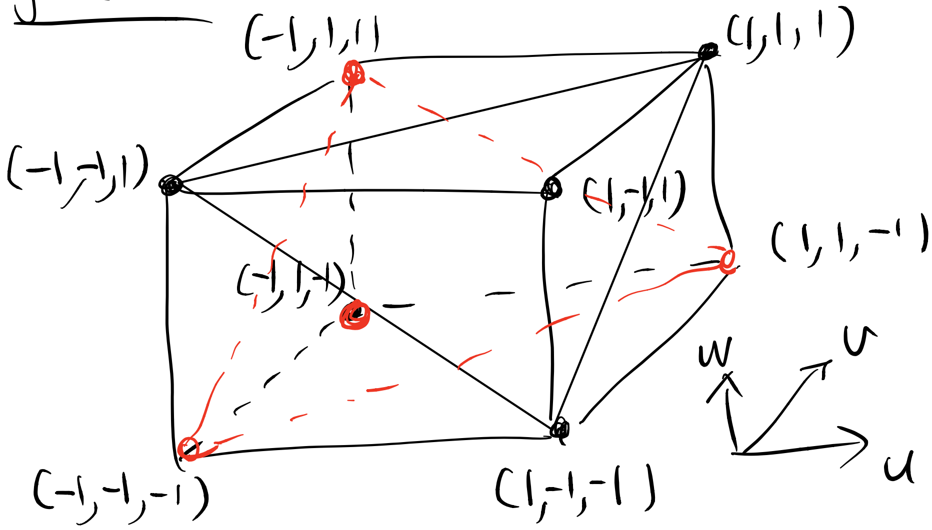
eg31 (cont'd)



$B = C$ by symmetry : $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \leftrightarrow \begin{pmatrix} -u \\ -v \\ -w \end{pmatrix}$

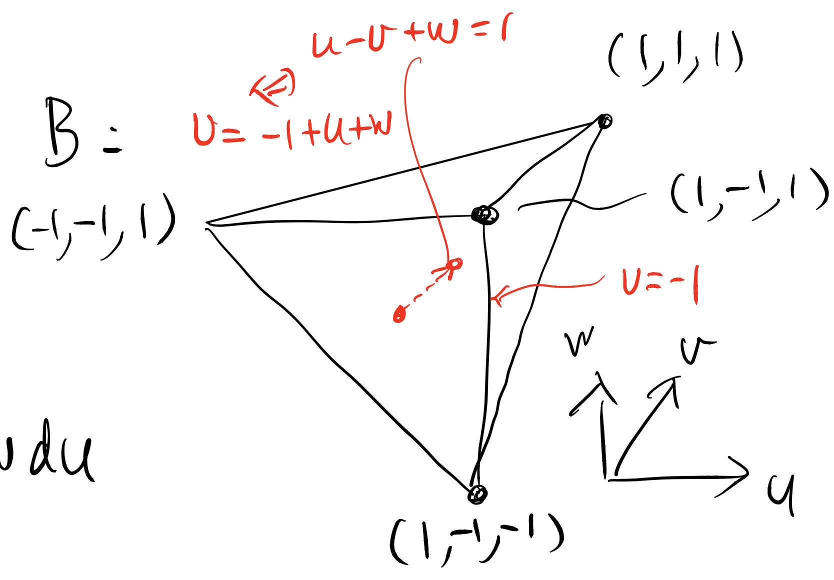
$$\det J = \left| \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \right| = -1$$

$$\left| \frac{\partial(\cdots)}{\partial(\cdots)} \right| = 1$$

$$u - v + w = 1 \quad \leftrightarrow \quad u - v + w = -1$$

$$\& \quad u^4 \quad \leftrightarrow \quad (-u)^4$$

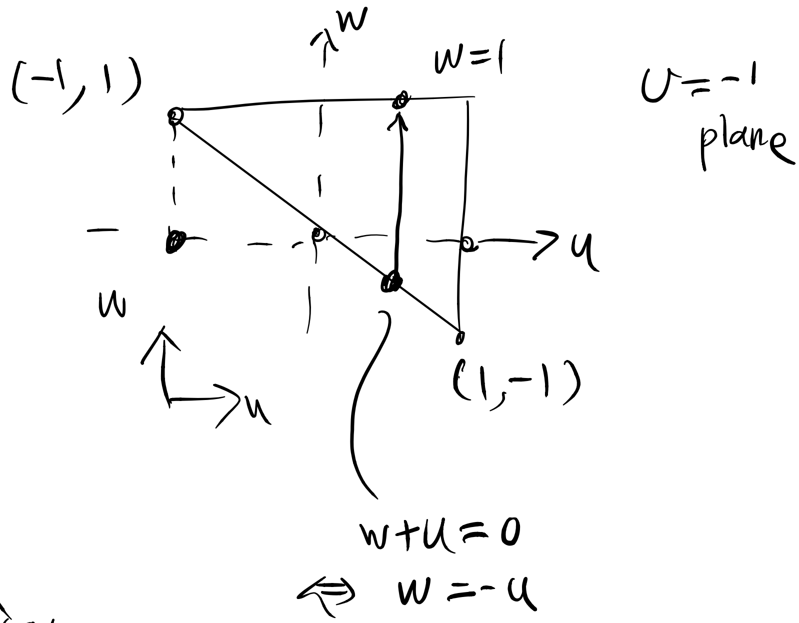
So we only need to do B :



$$\therefore \quad B = \int_{-1}^{1} \int_{-u}^{1} \int_{-1}^{-1+u+w} u^4 \, dv \, dw \, du$$

$$= \frac{3}{35}$$

(easily check!)



This completes the calculation.

Vector Analysis

Notation: Usually in textbooks, vectors are denoted by boldface $\mathbf{i}$, but hard to do it on screen, so my notation for vectors is:

$$\begin{cases} \text{general vector: } \vec{v}, \vec{F}, \vec{r}, \vec{\nabla}, \ldots \\ \text{unit vector: } \hat{i}, \hat{j}, \hat{k}, \hat{n} \end{cases}$$

Line integrals in $\mathbb{R}^3$ ($\mathbb{R}^n$)
(path integrals)

Def 9 The line integral of a function f on a curve (path, line) C with parametrization

$$\vec{r}: [a,b] \to \mathbb{R}^3$$
$$t \mapsto \vec{r}(t) = (x(t), y(t), z(t))$$

is $\displaystyle\int_C f(\vec{r})\, ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\vec{r}(t_i))\, \Delta s_i$

if the limit exists.

where $\begin{cases} P \text{ is a partition of } [a,b], \text{ and} \\ \Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} \end{cases}$

(i.e. ds = length element of a curve.)

Remarks:

(1) If $f \equiv 1$, $\int_C 1 \, ds$ = arc length of C.

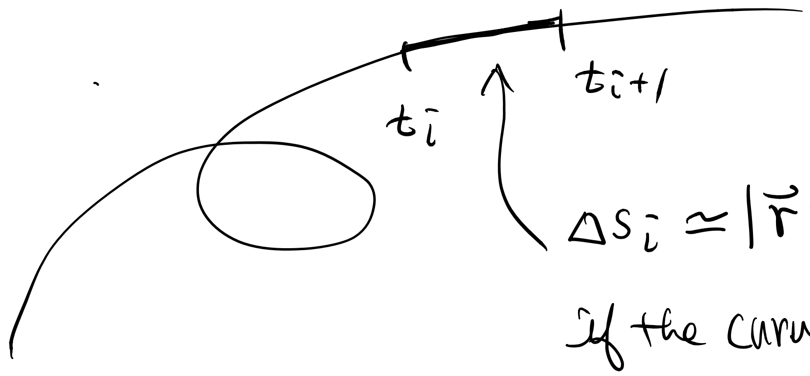
(2) The definition is well-defined, i.e. the RHS in the definition is independent of the parametrization $r(t)$.

Def 9′ (Formula for line integral)

Notation as in Def 9, then

$$\int_C f(\vec{r}) \, ds = \int_a^b f(\vec{r}(t)) \, |\vec{r}\,'(t)| \, dt$$

where $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$



$\Delta s_i \simeq |\vec{r}'(t_i)\Delta t_i| = |\vec{r}'(t_i)|\,\Delta t_i$

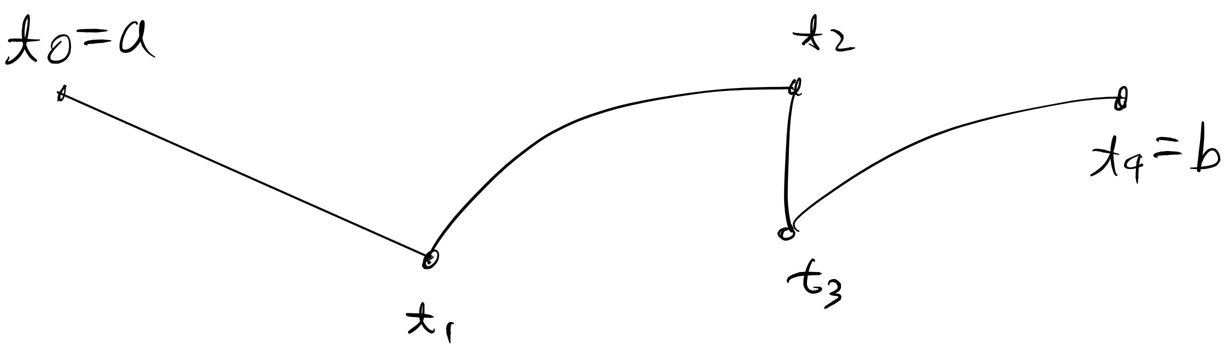
if the curve is differentiable.

<u>Notes:</u> (1) If $\vec{r}(t)$ is only piecewise differentiable, then the RHS of <u>Def 9'</u> becomes sum of each pieces:

If $[a,b] = [t_0, t_1] \cup \cdots \cup [t_{k-1}, t_k]$

s.t. $\vec{r}\big|_{[t_i, t_{i+1}]}$ is differentiable,

then $\displaystyle\int_C f(\vec{r})\,ds = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f(\vec{r})\,|\vec{r}'(t)|\,dt$.



(2) "$ds = |\vec{r}'(t)|\,dt$" is usually referred as the <u>arc length element</u>.

eg 32 : $f(x,y,z) = x - 3y^2 + z$

C = line segment joining the origin and $(1,1,1)$.

Find $\int_C f(x,y,z)\,ds$.

Soln : Parametrize C by

$$\vec{r}(t) = t(1,1,1) = (t,t,t), \quad t \in [0,1]$$

(ie. $x(t)=t$, $y(t)=t$, $z(t)=t$)

Then $\vec{r}'(t) = (1,1,1)$, $\forall t \in [0,1]$

$\Rightarrow \quad |\vec{r}'(t)| = \sqrt{3}$.

and $$\int_C f(x,y,z)\,ds = \int_0^1 f(t,t,t)\sqrt{3}\,dt$$ (Check!)

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt = 0.$$

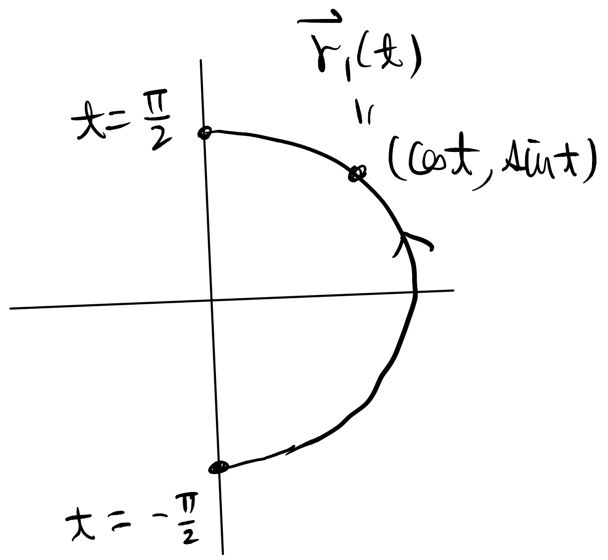
#

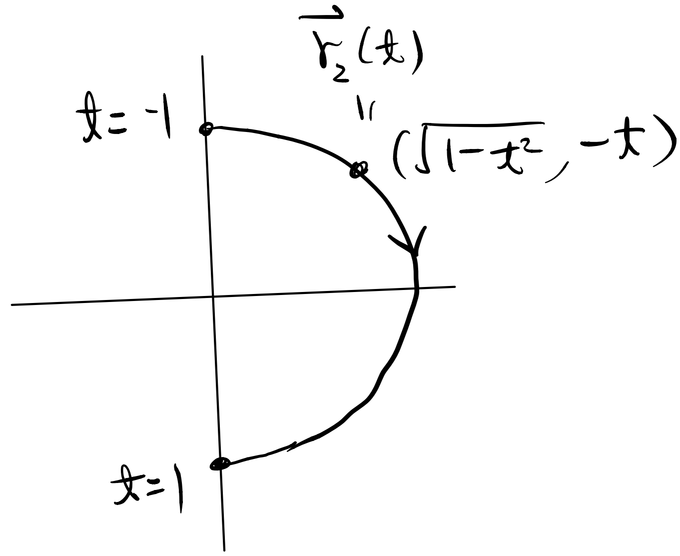
eg 33 : Let C be curve in $\mathbb{R}^2$ (ie. $z(t) \equiv 0$)

and its has 2 parametrizations :

$\vec{r}_1(t) = (\cos t, \sin t)$, $\quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$\vec{r}_2(t) = (\sqrt{1-t^2}, -t)$, $\quad -1 \le t \le 1$





Suppose $f(x,y)=x$. Find $\int_C f(x,y)\,ds$.

Soln: (i) $\vec{r}_1(t)=(\cos t, \sin t)$

$\Rightarrow \vec{r}_1'(t)=(-\sin t, \cos t)$

$\Rightarrow |\vec{r}_1'(t)|=1$

$$\int_C f(x,y)\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\vec{r}_1(t))\,|\vec{r}_1'(t)|\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot 1 \cdot dt = 2.$$

(ii) For $\vec{r}_2(t)=\left(\sqrt{1-t^2}, -t\right)$, $-1\le t\le 1$

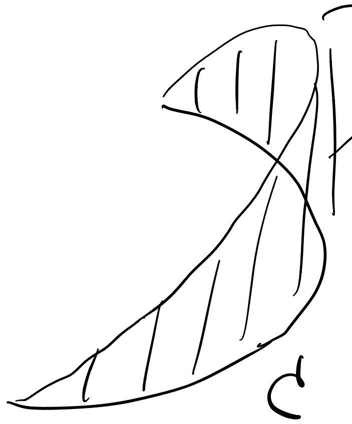
$$\int_C f(x,y)\,ds = \int_{-1}^{1} \sqrt{1-t^2}\cdot\left|\left(\frac{-t}{\sqrt{1-t^2}}, -1\right)\right|dt$$

$$= \int_{-1}^{1} \sqrt{1-t^2} \sqrt{\left(\frac{t}{\sqrt{1-t^2}}\right)^2+1} \, dt$$

$$= \int_{-1}^{1} dt = 2 \qquad \text{(check!)}$$

Same answer even with different directions of the parametrizations. ※

("ds" = arc length which has no direction.)



graph of f over C

$\int_C f\,ds$ = (signed) area under the graph of f over C.

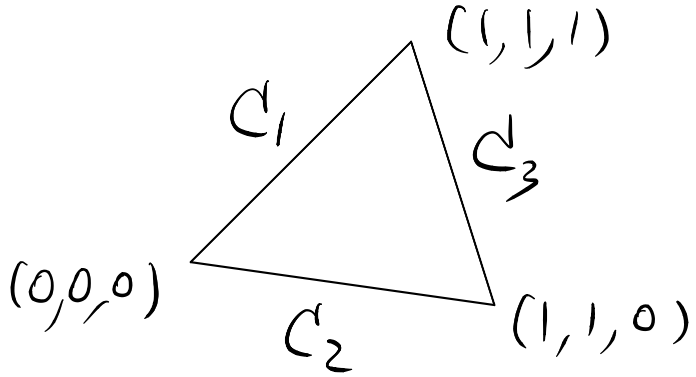
Prop 7: If C is a piecewise smooth curve made by joining $C_1, C_2, \cdots, C_n$ end to end, then

$$\int_C f\,ds = \sum_{i=1}^{n} \int_{C_i} f\,ds$$

(Pf: Clear from the remark of def 9'.)

eg 34: Let $f(x,y,z) = x - 3y^2 + z$ (again)

C_1, C_2, C_3 are as in the figure:



We already did $\int_{C_1} f\,ds$ and $\int_{C_1} f\,ds = 0$.

One can do $\int_{C_2 \cup C_3} f\,ds = \int_{C_2} f\,ds + \int_{C_3} f\,ds$

(can be calculated similarly as in the C_1 case.)

$= -\frac{\sqrt{2}}{2} - \frac{3}{2}$ (Ex: check!)

The point is: $\int_{C_1} f\,ds = 0 \neq -\frac{\sqrt{2}+3}{2} = \int_{C_2 \cup C_3} f\,ds$

even C_1 & $C_2 \cup C_3$ have the same end points!

Conclusion:

Line integral of a function depends not only on the end points, but also the path.

Vector Fields

Def 10: Let $D \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be a region, then a vector field on D is a mapping

$$\vec{F} : D \to \mathbb{R}^2 \text{ or } \mathbb{R}^3 \text{ respectively.}$$

In component form:

$\mathbb{R}^2$: $\vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$

$\mathbb{R}^3$: $\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + L(x,y,z)\,\hat{k}$

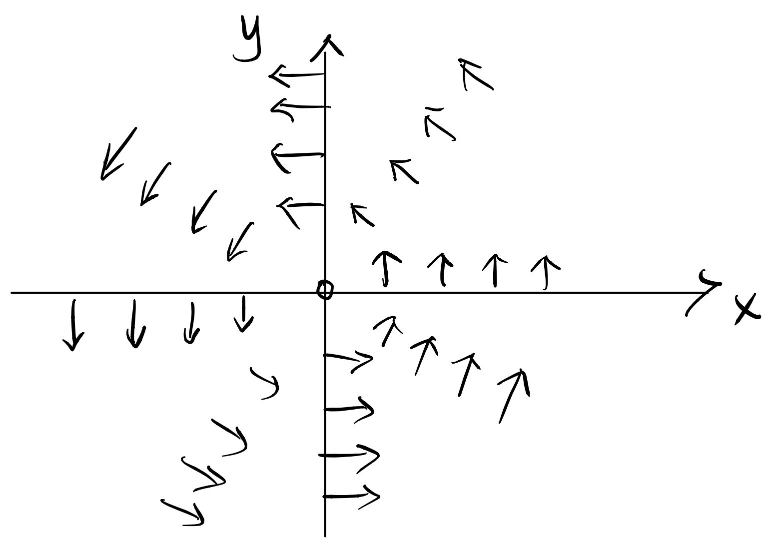
where M, N, L are functions on D called the <u>components</u> of $\vec{F}$.

<u>eg 35</u>: $\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$

$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

(in polar coordinates)

$$\Rightarrow \begin{cases} |\vec{F}(x,y)| = 1 \\ \vec{F}(x,y) \perp \vec{r} = (x,y) = r(\cos\theta, \sin\theta) \end{cases}$$



(Ex: Sketch $\vec{F}(x,y) = x\hat{i} + y\hat{j} = \vec{r}$)

eg36 (Gradient vector field of a function)

(i) $f(x,y) = \frac{1}{2}(x^2+y^2)$

$$\vec{\nabla} f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (x,y) = x\hat{i} + y\hat{j} = \vec{r}$$

(ii) $f(x,y,z) = x$

$$\vec{\nabla} f = (1,0,0) = \hat{i}.$$

eg37: Let C be a curve in $\mathbb{R}^2$ parametrized by

$$\vec{r} : [a,b] \to \mathbb{R}^2$$
$$t \mapsto (x(t), y(t)) = \vec{r}(t)$$

Recall: $\hat{T}$ = unit tangent vector field along C

$$= \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Note that this vector field only defined on C, but outside C.