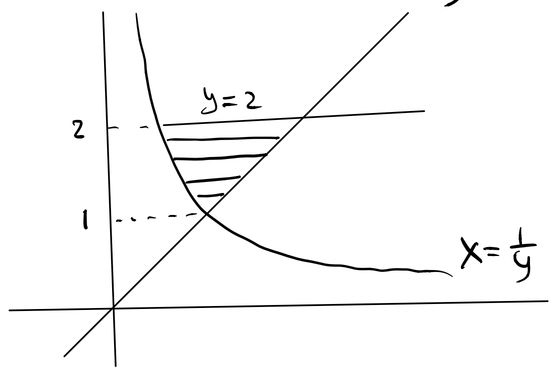
eg30 $\quad I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$

Domain for integration:

Observation:



$x=y \leftrightarrow \sqrt{\frac{y}{x}}=1$

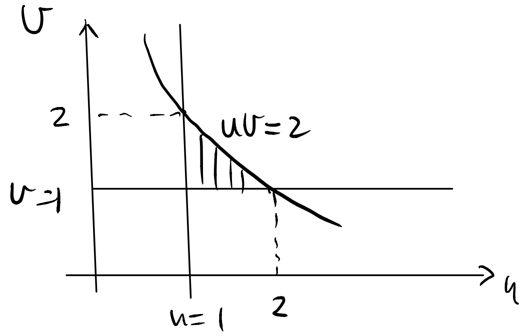
$x=\frac{1}{y} \leftrightarrow \sqrt{xy}=1$

Let $u=\sqrt{xy}$, $v=\sqrt{\frac{y}{x}}$

Then $x=y \leftrightarrow v=1$

$x=\frac{1}{y} \leftrightarrow u=1$

$y=2 \leftrightarrow uv=2$



And the Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$

$= \frac{2u}{v}$ (check!)

$$I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

$$= \int_1^2 \int_1^{\frac{2}{u}} v e^u \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du$$ (by change of variable famula)

$$= \int_1^2 \int_1^{\frac{2}{u}} 2ue^u \, dv\, du$$ (note: $\frac{2u}{v} > 0$)

$$= \int_1^2 \left[2ue^u \left(\int_1^{\frac{2}{u}} dv \right) \right] du$$

$$= \int_1^2 (4e^u - 2ue^u)\, du$$ (check)

$$= 2e(e-2)$$ (check: by integration-by-parts)

Substitutions in triple integrals

$$\begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ \cup & & \cup \end{array}$$

$$\phi(u,v,w) = (x,y,z) \;:\; G \to D$$

with
$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$$

1-1, onto, differentiable & inverse differentiable too.

Def 8 Jacobian (determinant) of transformation in $\mathbb{R}^3$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \\ \frac{\partial k}{\partial u} & \frac{\partial k}{\partial v} & \frac{\partial k}{\partial w} \end{pmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Note: Chain rule $\Rightarrow$

$$\begin{cases} \text{2-dim}: & \dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(s,t)} = \dfrac{\partial(x,y)}{\partial(s,t)} \\[2ex] \text{3-dim}: & \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \dfrac{\partial(u,v,w)}{\partial(s,t,r)} = \dfrac{\partial(x,y,z)}{\partial(s,t,r)} \end{cases}$$

$$\Rightarrow \begin{cases} \text{2-dim}: & \dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\frac{\partial(x,y)}{\partial(u,v)}} \\[2ex] \text{3-dim}: & \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{\frac{\partial(x,y,z)}{\partial(u,v,w)}} \end{cases}$$

Thm 7: Under similar conditions of Thm 6

$$\iiint_G F(x,y,z)\,dxdydz$$
$$= \iiint_D F(\phi(x,y,z))\,|J(u,v,w)|\,dudvdw$$

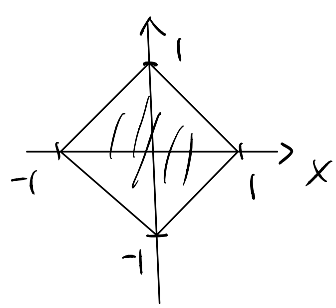
$F(g(u,v,w), h(u,v,w), k(u,v,w))$

eg31: Let $D = \{(x,y,z) \in \mathbb{R}^3 : |x|+|y|+|z| \le 1\}$

Evaluate $\iiint_D (x+y+z)^4\,dV$.

Soln If $z=0$, $|x|+|y| \le 1$

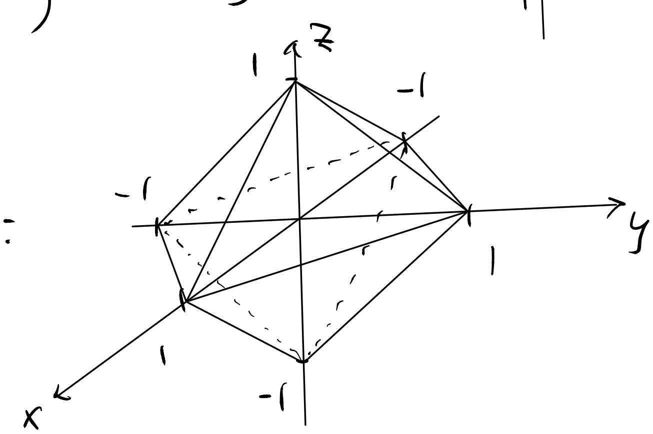
bounded by $\pm x \pm y = 1$



Hence D looks like

bounded by 8 planes:

$\pm x \pm y \pm z = 1$



To make ∂D and $(x+y+z)^4$ simpler.

Let $\begin{cases} u = x+y+z \\ v = x+y-z \\ w = x-y-z \end{cases} \longleftrightarrow \begin{cases} x = \frac{1}{2}(u+w) \\ y = \frac{1}{2}(v-w) \\ z = \frac{1}{2}(u-v) \end{cases}$

$$\Rightarrow \quad J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{4}$$

Boundary planes:

$$\pm x \pm y \pm z = 1 \longleftrightarrow u = \pm 1, \quad v = \pm 1, \quad w = \pm 1$$

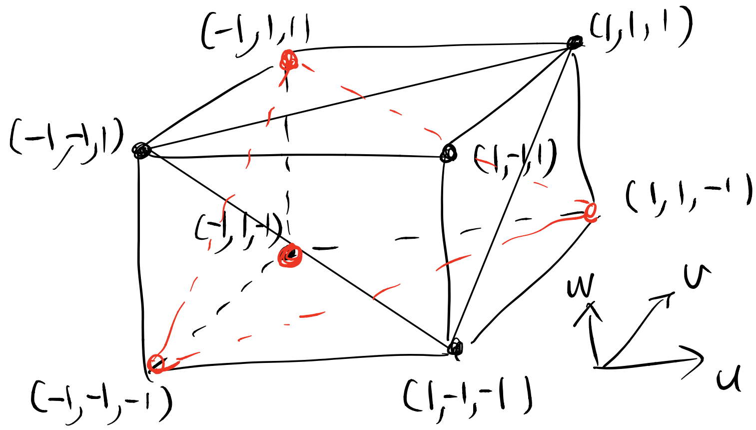
$$\left(\pm(x-y+z) = 1 \right) \Leftrightarrow u - v + w = \pm 1 \quad \text{(check!)}$$

Substitution $\Rightarrow$

$$\iiint_D (x+y+z)^4 dV = \iiint_{\substack{-1 \le u,v,w \le 1 \\ -1 \le u-v+w \le 1}} u^4 \, |J(u,v,w)| \, dv\,dw\,du$$

$$(A =) = \iiint_{-1 \le u,v,w \le 1} \frac{u^4}{4} dv\,dw\,du - \underbrace{\iiint_{\substack{-1 \le u,v,w \le 1 \\ u-v+w \ge 1}} \frac{u^4}{4} dv\,dw\,du}_{(B)}$$

$$- \iiint_{-1 \le u,v,w \le 1,\ u-v+w \le -1} \frac{u^4}{4} dv\,dw\,du = (C)$$



	$u-v+w$
(1,-1,1)	3
(1,1,1) (-1,-1,1) (1,-1,-1)	1
(-1,1,1) (-1,-1,-1) (1,1,-1)	-1
(-1,1,-1)	-3

Then

$$A = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{u^4}{4}\, dv\, dw\, du = \frac{2}{5}$$

$$B = \int_{-1}^{1}\int_{-u}^{1}\int_{-1}^{-1+u+w} \frac{u^4}{4}\, dv\, dw\, du = \frac{3}{35}$$

$$C = \int_{-1}^{1}\int_{-1}^{-u}\int_{1+u+w}^{1} \frac{u^4}{4}\, dv\, dw\, du = \frac{3}{35}$$

(next time)

$$\Rightarrow \iiint_D (x+y+z)^4\, dx\, dy\, dz = \frac{2}{5} - \frac{3}{35} - \frac{3}{35} = \frac{8}{35}$$

(check!).