Change of Variables formula

(Substitution in multiple integral)

Review of 1-variable

$$\int_a^b f(x)\,dx = \int_c^d \left[f(x(u))\frac{dx}{du}\right] du$$

$x = x(u)$ for $u \in [c,d]$

provided $\frac{dx}{du} > 0$ $(\Rightarrow c < d)$

and $\int_a^b f(x)\,dx = \int_d^c f(x(u))\frac{dx}{du}\,du$ if $\frac{dx}{du} < 0$

$(\Rightarrow c > d)$

Recall, in Riemann sum (of general dimensions):

$$\int_{[a,b]} f(x)\,dx$$

← limiting form of $|\Delta x|$ (unlike 1-variable Δx)

$[a,b]$ ← as set (we don't about the direction)

we actually have

$$\int_a^b f(x)\,dx = \begin{cases} \int_{[a,b]} f(x)\,dx, & \text{if } a \leq b \\ -\int_{[a,b]} f(x)\,dx, & \text{if } a \geq b \end{cases}$$

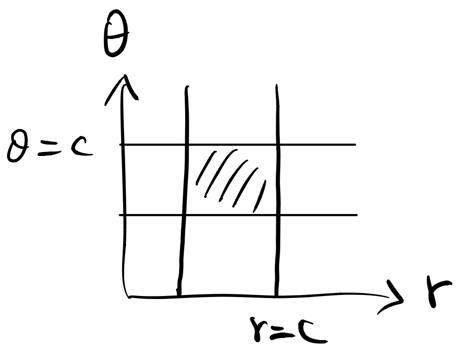
Combine these

$$\Delta x \xrightarrow{x = x(u)} \Delta u \qquad \frac{|\Delta u|}{|\Delta x|} \sim \left|\frac{dx}{du}\right|$$

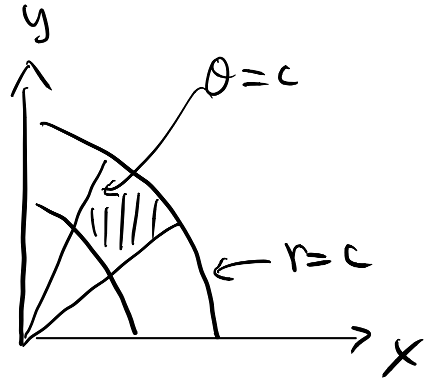
$$\int_{[a,b]} f(x)\,dx = \int_{[c,d]} f(x) \left|\frac{dx}{du}\right| du .$$

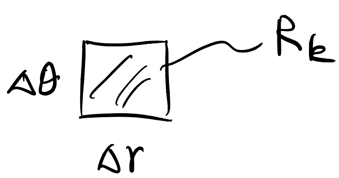
Back to multiple integrals:

<u>Recall</u>: polar coordinates

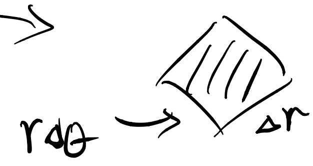


$$\xrightarrow{\ \phi\ } \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$





$\longrightarrow$ $\phi(R_k)$



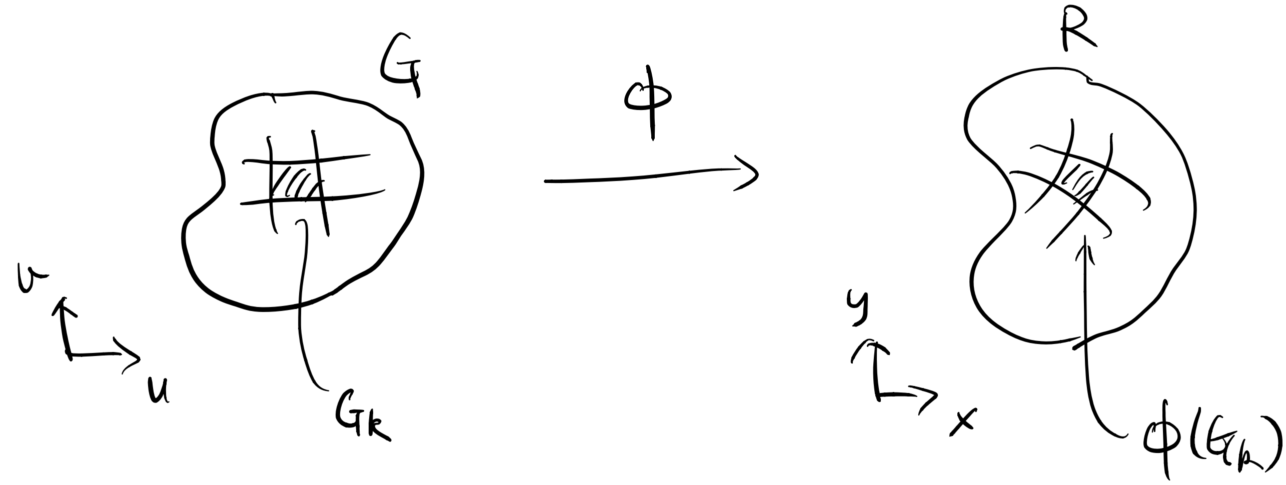
$$\Delta r \Delta\theta \cong \text{Area}(R_k) \qquad\qquad r\Delta r\Delta\theta \cong \text{Area}(\phi(R_k))$$

$$\frac{\text{Area}(\phi(R_k))}{\text{Area}(R_k)} \to r \quad \text{as "} R_k \to \text{point"}$$

General change of coordinates formula in $\mathbb{R}^2$

Suppose $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ denoted by $\phi : G \to R$ (1-1 onto)

$G \subset$ uv-plane, $R \subset$ xy-plane



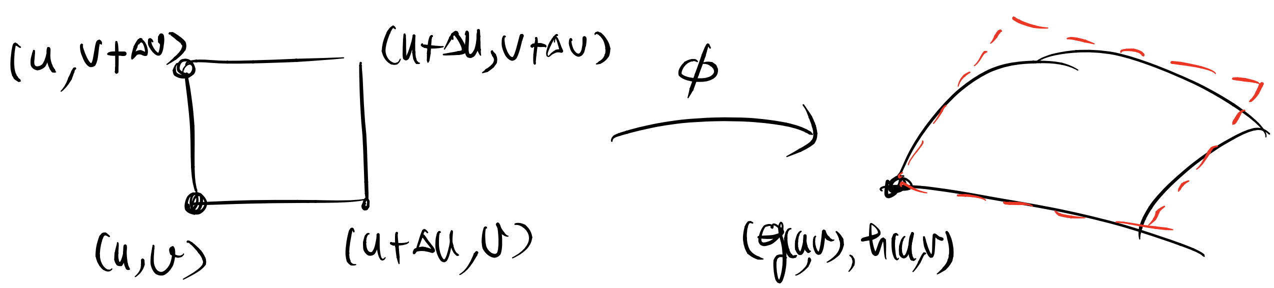
Idea: we need to find

$$\frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \to \,? \quad \text{as "} G_k \to \text{point"}$$

If ϕ is C^1-transformation (should be diffeomorphism: 1-1 onto & ϕ, ϕ^{-1} are C^1)

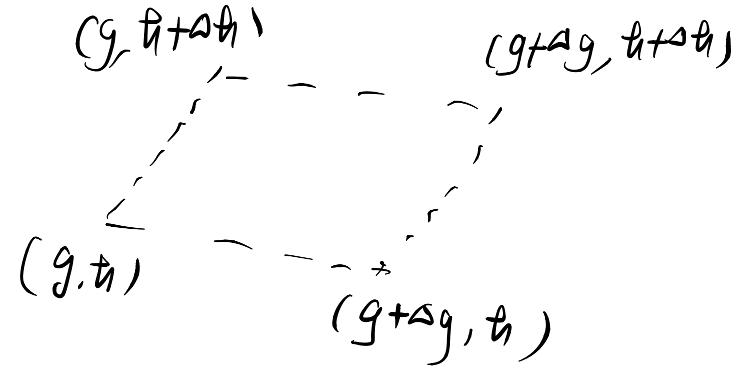
($C^1 \not\Rightarrow$ diffeo.)

$$\begin{cases} g(u+\Delta u, v+\Delta v) \cong g(u,v) + \frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v + \cdots \\ h(u+\Delta u, v+\Delta v) \cong h(u,v) + \frac{\partial h}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v + \cdots \end{cases}$$



$$\Delta g \sim \frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v$$

$$\Delta h \sim \frac{\partial h}{\partial u}\Delta u + \frac{\partial h}{\partial v}\Delta v$$



$$\begin{pmatrix} \Delta g \\ \Delta h \end{pmatrix} \cong \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}.$$

$$\Rightarrow \quad \frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \sim \frac{\Delta g \Delta h}{\Delta u \Delta v} \sim \left| \det \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} \right|$$

(by linear algebra)

<u>Def 7</u> Define the <u>Jacobian</u> $J(u,v)$ of the "coordinate" transformation $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

eg 29 (i) $x = r\cos\theta,\ y = r\sin\theta$

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = r \text{ (check!)}.$$

Hence we should have formula:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

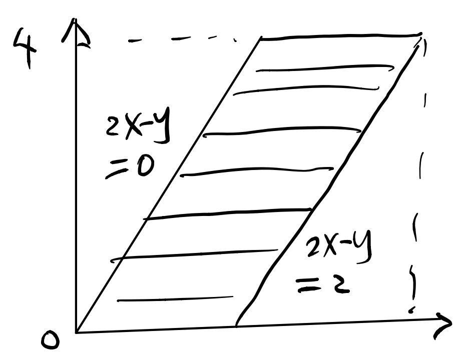
$$= \iint_G f(g(u,v), h(u,v)) \, |J(u,v)| \, du\,dv$$

<u>eg 29 (ii)</u> Hence in polar coordinates

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \iint_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta \quad \#$$

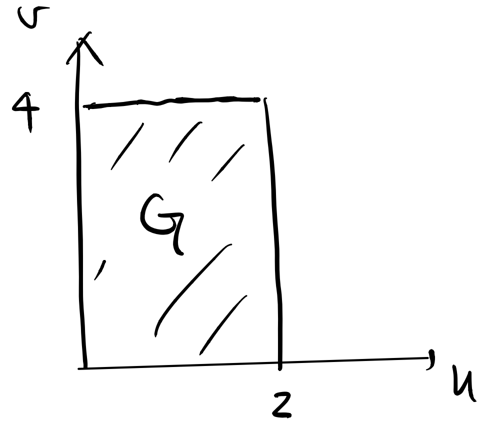
<u>eg 30</u> $\displaystyle\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$



lower limit $x = \frac{y}{2} \Leftrightarrow 2x-y=0$

upper limit $x = \frac{y}{2}+1 \Leftrightarrow 2x-y=2$

Define $\begin{cases} u = 2x - y \\ v = y \end{cases}$



Then $\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = v \end{cases}$

$\begin{cases} 2x - y = 0 \longleftrightarrow \\ 2x - y = 2 \longleftrightarrow \end{cases} \begin{cases} u = 0 \\ u = 2 \end{cases}$

$\begin{cases} y = 0 \\ y = 4 \end{cases} \longleftrightarrow \begin{cases} v = 0 \\ v = 4 \end{cases}$

$$J(u,v) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

$$\therefore \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2} \, dx \, dy$$

$$= \int_0^4 \int_0^2 \frac{u}{2} \left| \frac{1}{2} \right| du \, dv = 2 \quad (\text{Check!})$$

※

Thm 6 Suppose $\phi : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ is a diffeomorphism (1-1, onto; ϕ & ϕ^{-1} both diff.) mapping a region G (closed and bounded) in the uv-plane onto a region R (closed and bounded) in the xy-plane (except possibly on the boundary). Suppose $f(x,y)$ is continuous on R, then

$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_G f\circ\phi(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

Note: ϕ is diffeomorphism $\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \neq 0$.

Pf of Thm 6:

step 0: • We need better notations and terminology: we'll denote in this prove:

$$J(\phi) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$ the Jacobian matrix.

and $\frac{\partial(x,y)}{\partial(u,v)} = \det J(\phi)$ the Jacobian determinant.

- We also use "index" notations for variables (x_1, x_2) or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ (instead of (x,y), $\begin{pmatrix} x \\ y \end{pmatrix}$).

Step 1: Let $F: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix}$ near a point p

with $\dfrac{\partial(f_1,f_2)}{\partial(x_1,x_2)} \neq 0$ at p. Then, near the point p,

F can be decomposed as $F = H \circ K$

with H, K of the forms

$$K: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} k(x_1,x_2) \\ x_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} k(x_1,x_2) \\ x_1 \end{pmatrix}$$

and $$H: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ h(y_1,y_2) \end{pmatrix}$$

such that $\det J(K) \neq 0$ & $\det J(H) \neq 0$.

Pf of Step 1:

Case 1: $\dfrac{\partial f_1}{\partial x_1}(p) \neq 0$

Define $k(x_1,x_2) = f_1(x_1,x_2)$ near p.

Then the transformation

$$K = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ defined by}$$

$$\begin{cases} y_1 = f_1(x_1, x_2) \\ y_2 = x_2 \end{cases}$$

is of the required form and has Jacobian matrix

$$J(K) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \det J(K)(p) = \frac{\partial f_1}{\partial x_1}(p) \neq 0 .$$

By Inverse Function Theorem, K is invertible near p

and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = K^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} g(y_1, y_2) \\ y_2 \end{pmatrix}$ is diff. at $K(p)$

with $J(K^{-1})_{K(p)} \cdot J(K)_p = Id.$

i.e.

$$\begin{pmatrix} \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow \quad \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_1} = 1 \quad \& \quad \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_2} + \frac{\partial g}{\partial y_2} = 0 .$$

In particular, $\det J(K^{-1})_{K(p)} = \frac{1}{\det J(K)_p} \neq 0 .$

Now, define $h(y_1, y_2) = f_2\left(K^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)$

$$= f_2(g(y_1, y_2), y_2) \quad (= f_2(x_1, x_2))$$

and $H: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ by

$$\begin{cases} z_1 = y_1 \\ z_2 = h(y_1, y_2) \end{cases}$$

Then $J(H) = \begin{pmatrix} 1 & 0 \\ \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} \end{pmatrix}$

Note that $\displaystyle \frac{\partial h}{\partial y_2} = \frac{\partial f_2}{\partial x_1}\frac{\partial x_1}{\partial y_2} + \frac{\partial f_2}{\partial x_2}\frac{\partial x_2}{\partial y_2}$

$$= \frac{\partial f_2}{\partial x_1}\frac{\partial g}{\partial y_2} + \frac{\partial f_2}{\partial x_2}\frac{\partial x_2}{\partial y_2}$$

$$= \frac{\partial f_2}{\partial x_1}\left(-\frac{\partial f_1}{\partial x_2}\frac{\partial g}{\partial y_1}\right) + \frac{\partial f_2}{\partial x_2} \cdot 1$$

$$= -\frac{\frac{\partial f_2}{\partial x_1}\frac{\partial f_1}{\partial x_2}}{\frac{\partial f_1}{\partial x_1}} + \frac{\partial f_2}{\partial x_2}$$

$$= \frac{1}{\frac{\partial f_1}{\partial x_1}} \det J(K) \neq 0$$

$\therefore \det J(H) \neq 0$.

So, H & K satisfy the requirements and we have

$$H\circ K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = H\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ \tilde{f}(y_1, y_2) \end{pmatrix}$$

$$= \begin{pmatrix} k(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Case 2: $\frac{\partial f_1}{\partial x_1}(p) = 0$

Since $\det J(F) \neq 0$, then $\frac{\partial f_2}{\partial x_1}(p) \neq 0$

Interchange the variables $\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$,

then the new mapping $\tilde{F}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$ satisfies the condition in case 1. Applying case 1 to $\tilde{F}$, then interchanging back to x_1, x_2. ※

Step 2: Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k(x_1, x_2) \\ x_2 \end{pmatrix}$

be a diffeo. from region R_1 to $R_2 = K(R_1)$. Then $\forall$ function $f(y_1, y_2)$ on R_2

$$\iint_{R_2} f(y_1, y_2)\, dy_1 dy_2 = \iint_{R_1} f\circ K(x_1, x_2) \left|\det J(K)\right| dx_1 dx_2$$

$$= \iint_{R_1} f(k(x_1, x_2), x_2) \left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right| dx_1 dx_2$$

Pf: By additivity property of integrations and cutting R_1 (and correspondingly $R_2 = K(R_1)$) into small regions,

we may assume $R_1 = [a,b] \times [c,d]$.

For any fixed $y_2 = x_2$, $y_1 = k(x_1, x_2) = k(x_1, y_2)$

can be regarded as a transformation of 1-variable.

Note that $\dfrac{\partial y_1}{\partial x_1} = \dfrac{\partial k}{\partial x_1}$

and $$0 \neq \det J(K) = \det \begin{pmatrix} \frac{\partial k}{\partial x_1} & \frac{\partial k}{\partial x_2} \\ 0 & 1 \end{pmatrix} = \frac{\partial k}{\partial x_1}$$

(since K is a diffeo.)

& $$\left|\frac{\partial y_1}{\partial x_1}\right| = \left|\det J(K)\right| \neq 0.$$

Note also that R_2 is of special type:

$$\{c \le y_2 \le d,\ k(a,y_2) \le y_1 \le k(b,y_2)\} \quad \left(\frac{\partial y_1}{\partial x_1} > 0\right)$$

or $$\{c \le y_2 \le d,\ k(b,y_2) \le y_1 \le k(a,y_2)\} \quad \left(\frac{\partial y_1}{\partial x_1} < 0\right)$$

By Fubini's Theorem (assuming $\frac{\partial y_1}{\partial x_1} > 0$, the other is similar.)

$$\iint_{R_2} f(y_1,y_2)\,dy_1dy_2 = \int_c^d \left(\int_{k(a,y_2)}^{k(b,y_2)} f(y_1,y_2)\,dy_1 \right) dy_2$$

By change of variable formula in 1-dim. and note

$$\int_{k(a,y_2)}^{k(b,y_2)} f(y_1,y_2)\,dy_1 = \int_a^b f(k(x_1,y_2),y_2) \left|\frac{\partial y_1}{\partial x_1}\right| dx_1.$$

Hence

$$\iint_{R_2} f(y_1,y_2)\,dy_1dy_2 = \int_c^d \int_a^b f(k(x_1,y_2),y_2) \left|\frac{\partial y_1}{\partial x_1}\right| dx_1\,dy_2$$

$$= \int_c^d \int_a^b f(k(x_1,x_2),x_2)\,|\det J(k)|\,dx_1\,dx_2$$

※

This step2 also holds for $K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k(x_1,x_2) \\ x_1 \end{pmatrix}$

and $H\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ h(x_1,x_2) \end{pmatrix}$.

$\boxed{\text{Step3: If the change of variables formula holds for } F \,\&\, G, \text{ then it holds for } F\circ G.}$

Pf: Easily by $J(F\circ G) = J(F)J(G)$. (Ex!) ※

Final step: Combining steps 1-3, and using additivity property of integration, we've proved the Thm 6 for general change of variable formula. ※

(Actually, this applies to all dimension.)