eg 27 let $f(x,y,z) = \frac{1}{x^2+y^2+z^2} = \frac{1}{\rho^2}$

$g(x,y,z) = \frac{1}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{1}{\rho^3}$ (unbounded at $\rho = 0$)

over unit ball $B = \{ (\rho, \phi, \theta) : 0 \le \rho \le 1 \}$

"Improper Integral"

$$\lim_{\varepsilon \to 0} \iiint_{B \setminus B_\varepsilon} f(x,y,z)\, dx\,dy\,dz = \text{exists?}$$

where $B_\varepsilon = \{ (\rho, \phi, \theta) : 0 \le \rho \le \varepsilon \}$

$$\lim_{\varepsilon \to 0} \iiint_{B \setminus B_\varepsilon} f(x,y,z)\, dx\,dy\,dz$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_\varepsilon^1 \frac{1}{\rho^2} \cdot \left(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \right)$$

$$= \lim_{\varepsilon \to 0} \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\pi} \sin\phi \, d\phi \right) \left(\int_\varepsilon^1 d\rho \right)$$

$$= \lim_{\varepsilon \to 0} 4\pi \, (1 - \varepsilon) = 4\pi$$

Hence we said that $f = \frac{1}{x^2+y^2+z^2}$ is integrable (in sense of Improper Integral)

For $g(x,y,z) = \frac{1}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{1}{\rho^3}$

$$\Rightarrow \lim_{\varepsilon \to 0} \iiint_{B \setminus B_\varepsilon} g(x,y,z)\,dx\,dy\,dz$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_\varepsilon^1 \frac{1}{\rho^3} \left(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta\right)$$

$$= \lim_{\varepsilon \to 0} 4\pi \log \frac{1}{\varepsilon} = +\infty \quad \text{not exists}$$

Hence we said that $g = \frac{1}{(x^2+y^2+z^2)^{3/2}}$ is not integrable.

\#

(Question: determine all $\beta > 0$ such that

$f = \frac{1}{\rho^\beta}$ is integrable in $B \subset \mathbb{R}^3$ (Ex!))

Application of Multiple integrals (Thomas' Calculus §15.6)

In applications, we often use the following:

In 2-dim.: R is a region in $\mathbb{R}^2$ with density $\delta(x,y)$.

- First moment about y-axis:

$$M_y = \iint_R x\,\delta(x,y)\,dA$$

- First moment about x-axis:

$$M_x = \iint_R y\,\delta(x,y)\,dA$$

- Mass: $$M = \iint_R \delta(x,y)\,dA$$

- Center of Mass (Centroid):

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right).$$

In 3-dim, D solid region in $\mathbb{R}^3$ with $\delta(x,y,z)$

First moments

- about yz-plane:

$$M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$$

- about xz-plane:

$$M_{xz} = \iiint_D y\,\delta(x,y,z)\,dV$$

- about xy-plane:

$$M_{xy} = \iiint_D z\,\delta(x,y,z)\,dV$$

- Mass: $$M = \iiint_D \delta(x,y,z)\,dV$$

- Center of Mass (Centroid)

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$$

In 2-dim, R region in $\mathbb{R}^2$ with density $\delta(x,y)$

Moment of inertia

- about the x-axis $\quad I_x = \iint_R y^2 \delta(x,y)\,dA$
- about the y-axis $\quad I_y = \iint_R x^2 \delta(x,y)\,dA$
- about the line L

$$I_L = \iint_R r(x,y)^2 \delta(x,y)\,dA$$

where $r(x,y) =$ distance between (x,y) and L.

- about the origin $\quad I_0 = \iint_R (x^2+y^2)\delta(x,y)\,dA$

In 3-dim. D solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

Moments of Inertia

- around x-axis : $I_x = \iiint_D (y^2+z^2)\,\delta(x,y,z)\,dV$

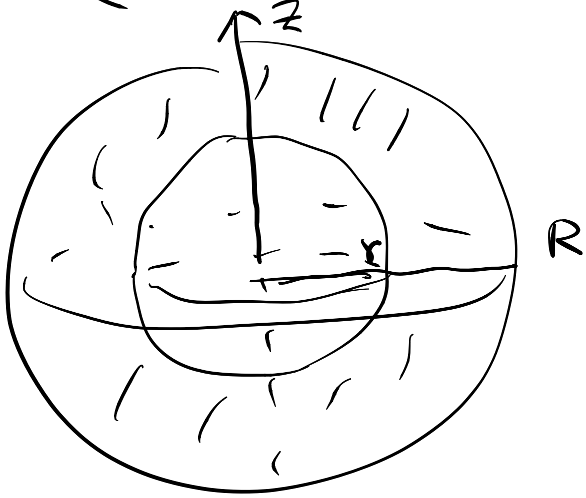
- around y-axis : $I_y = \iiint_D (x^2+z^2)\,\delta(x,y,z)\,dV$

- around z-axis : $I_z = \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV$

- around line L : $I_L = \iiint_D r(x,y,z)^2\,\delta(x,y,z)\,dV$

  where $r(x,y,z)$ = distance between (x,y,z) and L.

eg 28: Consider $D: r^2 \leq x^2+y^2+z^2 \leq R^2$ with density $\delta(x,y,z) \equiv \delta$. Express I_z in term of m = mass of D $(= \delta \mathrm{Vol}(D))$ r and R.



Soln: $I_z = \iiint_D (x^2+y^2) \cdot 1 \, dV$

$$= \int_0^{2\pi} \int_0^{\pi} \int_r^R (\rho^2 \sin^2\phi)(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta)$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin^3\phi \, d\phi\right)\left(\int_r^R \rho^4 \, d\rho\right)$$

$$= (2\pi)\left(\frac{4}{3}\right)\left(\frac{R^5-r^5}{5}\right)$$

$$= \frac{8\pi}{15}(R^2-r^2)\delta$$

Mass $m = \delta \operatorname{Vol}(\Omega) = \frac{4\pi}{3}(R^3 - r^3)\,\delta$

$$\Rightarrow \boxed{I_z = \frac{2m}{5}\,\frac{R^5 - r^5}{R^3 - r^3}} \quad (\text{true for } \delta = 1)$$

There are two interesting special cases:

(i) $r \to 0$, i.e. the whole solid ball

$$\Rightarrow \boxed{I_z = \frac{2m}{5} R^2}$$

(ii) $r \to R$, i.e. a (hollow) sphere made of infinitesimally thin sheet.

$$\Rightarrow \boxed{\begin{aligned} I_z &= \lim_{r \to R} \frac{2m}{5}\,\frac{R^5 - r^5}{R^3 - r^3} \\ &= \frac{2m}{5} \cdot \frac{5R^4}{3R^2} = \frac{2mR^2}{3} \end{aligned}}$$

Moment of inertia of hollow sphere

$>$ the solid ball (assuming the same mass)