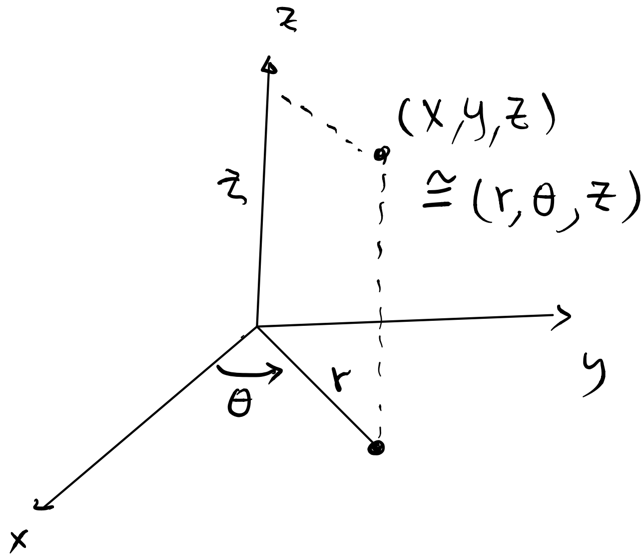
Cylindrical Coordinates in $\mathbb{R}^3$



- (r,θ) = polar coordinates for the xy-plane ($r \geq 0$)
- z = rectangular vertical coordinate.

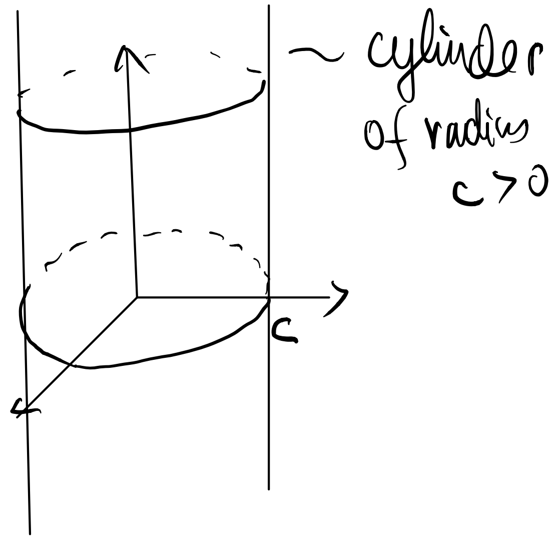
Then a point $P = (x,y,z)$ can be represented by (r,θ,z) where

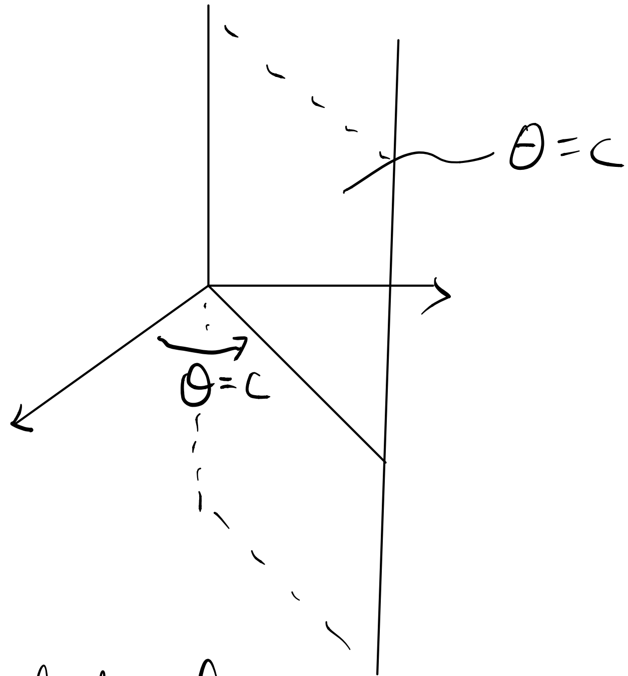
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

And (r,θ,z) is called the cylindrical coordinates for $\mathbb{R}^3$.

<u>Remark 1</u>: (c is a constant)

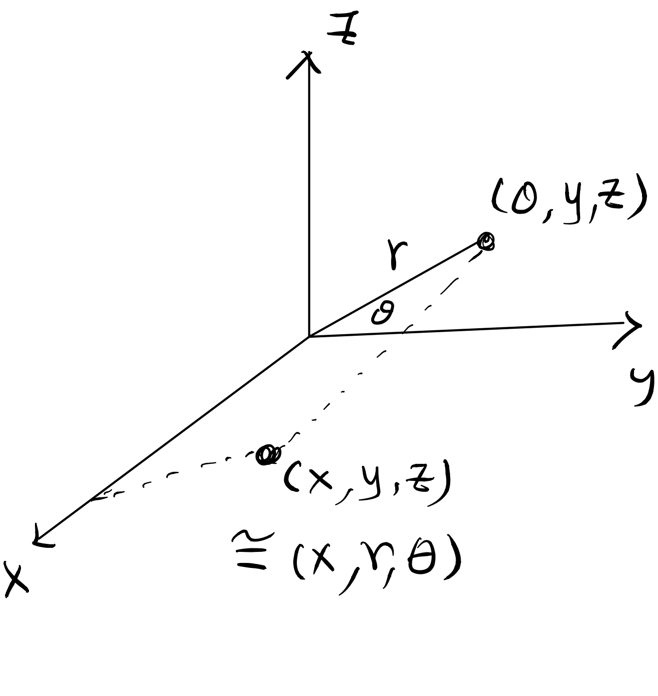
- $r = c$ ($c > 0$) describes a <u>cylinder</u>
- $\theta = c$ describes a half-plane
- $z = c$ describes a horizontal plane (as in rectangular coordinates)



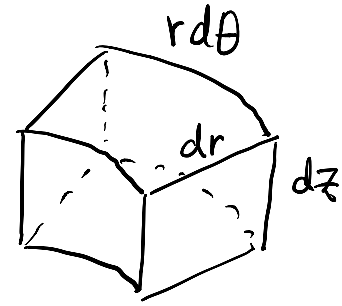


Remark 2 We can define cylindrical coordinates in other directions: eg

$$\boxed{\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}}$$



Volume element:

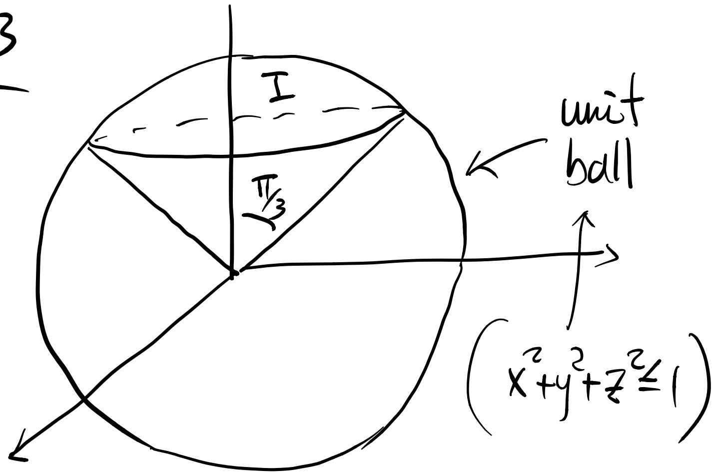


$$dV = dx\,dy\,dz$$

$$= r\,dr\,d\theta\,dz$$

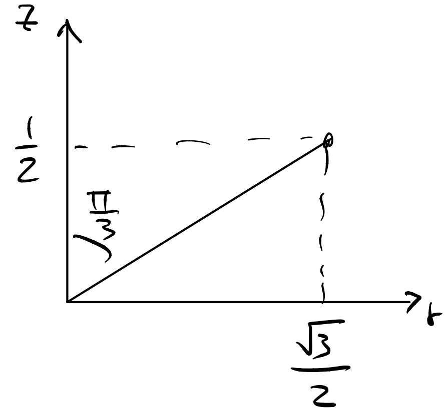
(order of the integration can be changed)

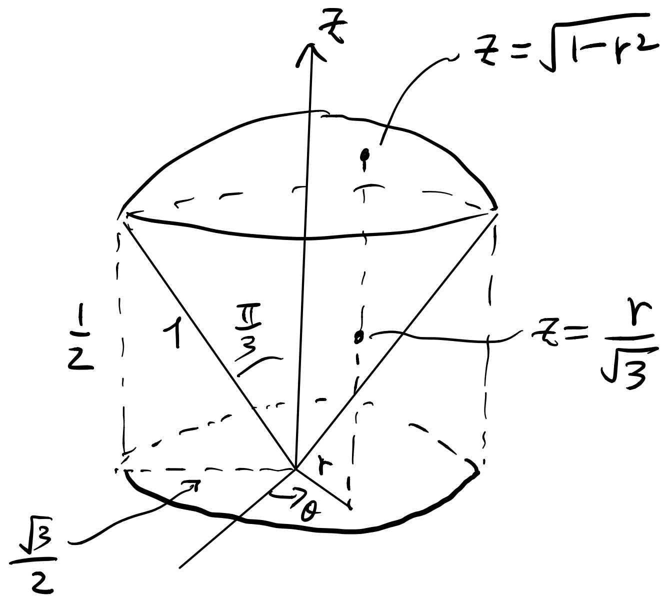
eg 23



Find the volume of the Ice-cream cone I given in the figure.

Soln :



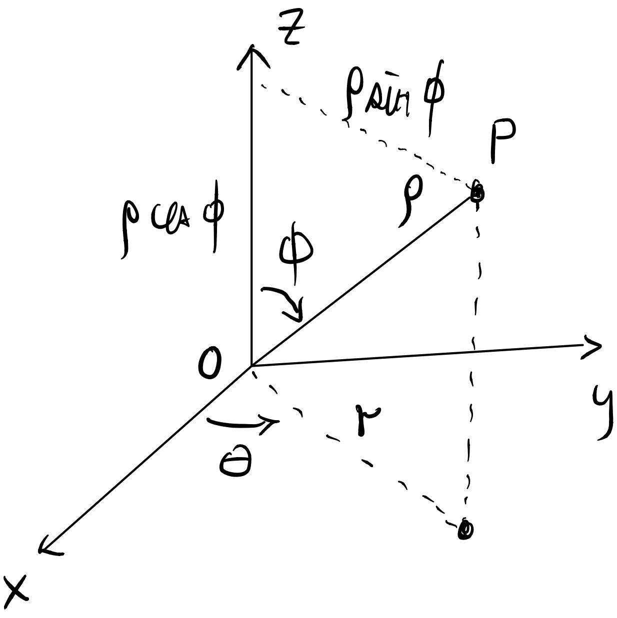


$$\mathrm{Vol}(I) = \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^{\frac{\sqrt{3}}{2}} r \cdot \left[\sqrt{1-r^2} - \frac{r}{\sqrt{3}}\right] dr$$

$$= \frac{\pi}{3} \quad \text{(check!)}$$

Spherical Coordinates in $\mathbb{R}^3$



(ρ, ϕ, θ) where

- ρ = distance from the origin $(\rho \geq 0)$
- ϕ = angle from the positive z-axis to $\overline{OP}$ $(0 \leq \phi \leq \pi)$
- θ = angle from cylindrical coordinates. $(0 \leq \theta \leq 2\pi)$

Remark: If (r, θ, z) is the cylindrical coordinates of P,

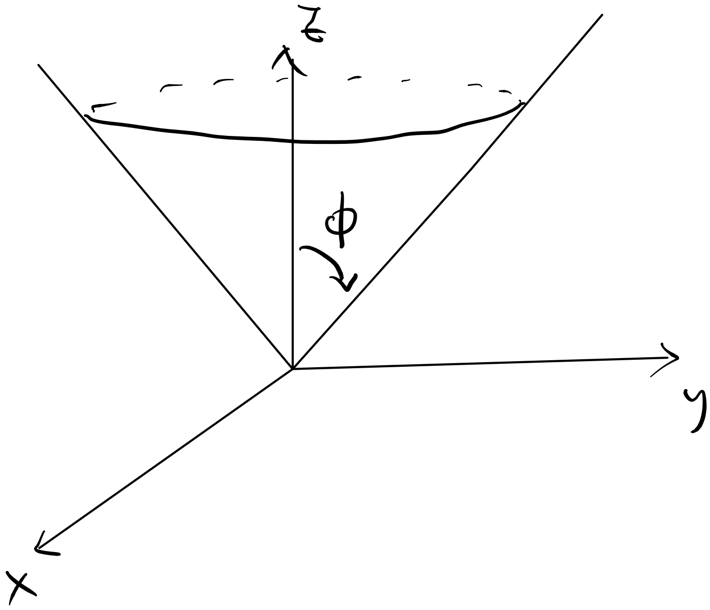
then $\begin{cases} r = \rho \sin\phi \\ z = \rho\cos\phi \end{cases}$.

Inparticular $z^2 + r^2 = \rho^2$

Then
$$\boxed{\begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \\ z = z = \rho\cos\phi \end{cases}}$$

Remark: If c is a constant, then

- $\rho = c$ $(c > 0)$ describes a sphere of radius c
- $\phi = c$ describes $\begin{cases} \text{+ve } z\text{-axis} & \text{if } c = 0 \\ \text{-ve } z\text{-axis} & \text{if } c = \pi \\ xy\text{-plane} & \text{if } c = \frac{\pi}{2} \\ \text{cone} & \text{otherwise} \end{cases}$
- $\theta = c$ describes a half-plane (vertical)

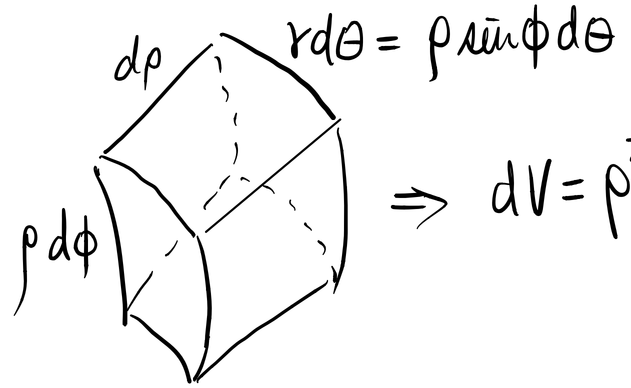


Volume element

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$= (\rho \sin\phi)(\rho\,d\rho\,d\phi)\,d\theta$$

$$= \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$



$\Rightarrow dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

eg24 Convert the following into spherical coordinates:

(1) $x^2+y^2+(z-1)^2=1$ (sphere)

(2) $z=-\sqrt{x^2+y^2}$ (cone)

Soln: (1) Sub. $\begin{cases} x=\rho\sin\phi\cos\theta \\ y=\rho\sin\phi\sin\theta \\ z=\rho\cos\phi \end{cases}$

into $x^2+y^2+(z-1)^2=1$

$$\Rightarrow \underbrace{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}+(\rho\cos\phi-1)^2=1$$

$$\Rightarrow \rho^2\sin^2\phi+\rho^2\cos^2\phi-2\rho\cos\phi+1=1$$

$$\Rightarrow \rho^2-2\rho\cos\phi=0$$

i.e. $\rho(\rho-2\cos\phi)=0$ together with $\rho\geq 0$

$$\Rightarrow \rho=2\cos\phi.$$

(2) Sub. the formula into $z=-\sqrt{x^2+y^2}$

$$\Rightarrow \quad \rho\cos\phi = -\rho\sin\phi \quad \left(\begin{array}{l}\rho\geq 0,\\ 0\leq\phi\leq\pi,\ \sin\phi\geq 0\end{array}\right)$$

For $\rho\neq 0$ (i.e. not the origin),

$$\cos\phi = -\sin\phi \quad (\text{i.e. } \tan\phi=-1)$$

$$\Rightarrow \quad \phi = \frac{3\pi}{4}.$$

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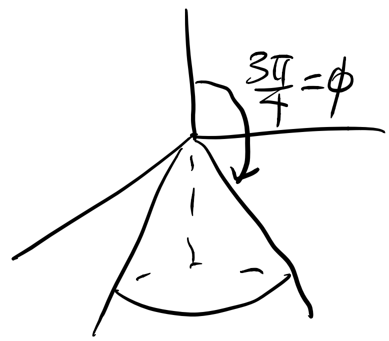
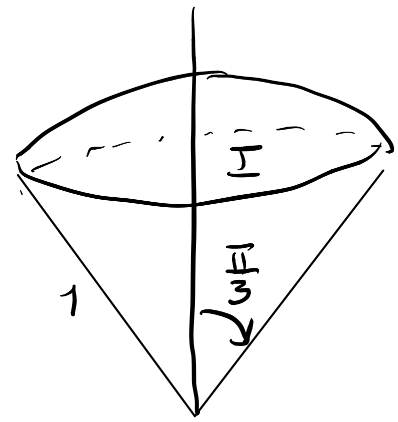


<u>eg 25</u> (see eg 23)

Volume of ice-cream cone again by spherical coordinates



<u>Soln</u>: The ice-cream cone I is given by

$$\begin{cases} 0\leq\rho\leq 1\\ 0\leq\phi\leq\frac{\pi}{3}\\ 0\leq\theta\leq 2\pi\end{cases}$$

$\Rightarrow \quad Vol(I) = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \underbrace{\rho^2 \sin\phi}_{} \, d\rho \, d\phi \, d\theta$

($\leftarrow$ don't miss this)

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{3}} \sin\phi \, d\phi\right)\left(\int_0^1 \rho^2 d\rho\right)$$

$$= \frac{\pi}{3} \quad \text{(check!)}$$

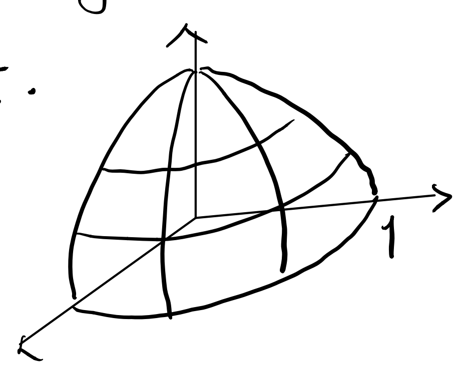
(must easier than cylindrical coordinates)

eg 26

$$f(x,y,z) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$$

(f is continuous, in fact value of f at one point doesn't affect $\iiint_D f \, dV$)

Let D = unit ball centered at origin intersecting with the first octant.



Then D can be represented in spherical coordinates:

$$\begin{cases} 0 \le \rho \le 1 \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

And (for $(x,y,z) \neq (0,0,0)$)

$$f(x,y,z) = \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \frac{(\rho \sin\phi)^2}{\rho}$$

$$= \rho \sin^2\phi .$$

(Note $f \to 0$ as $\rho \to 0$, so this eqt. actually valid even for $(x,y,z) = (0,0,0)$)

Hence

$$\iiint_D f(x,y,z)\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 (\rho \sin^2\phi)\, \underbrace{\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}_{dV}$$

$$= 2\pi \left(\int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi \right)\left(\int_0^1 \rho^3 d\rho \right)$$

$$= \frac{\pi}{12} \quad (\text{check!})$$

Hence the average of f over D

$$= \frac{\iiint_D f \, dV}{Vol(D)} = \frac{\frac{\pi}{12}}{\frac{1}{8} \cdot \frac{4\pi}{3}} = \frac{1}{2}$$

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