Triple Integrals

Def5 Let $f(x,y,z)$ be a function defined on a (closed and bounded) rectangular box

$$B = [a,b] \times [c,d] \times [r,s]$$

Then the triple integral of f over the box B is

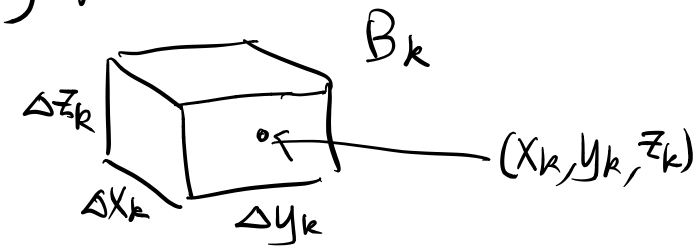
$$\iiint\limits_B f(x,y,z)\,dV = \lim_{\|P\| \to 0} \sum_k f(x_k, y_k, z_k)\,\Delta V_k$$

if this exists,

where (i) $P = P_1 \times P_2 \times P_3$ is a subdivision of B into sub-rectangular boxes by partitions P_1, P_2, P_3 of $[a,b]$, $[c,d]$, $[r,s]$ respectively, and

$$\|P\| = \max\left(\|P_1\|, \|P_2\|, \|P_3\|\right)$$

(ii) (x_k, y_k, z_k) is an arbitrary point in a sub-rectangular box B_k



(iii) $\Delta V_k = \mathrm{Vol}(B_k)$

$$= \Delta x_k \, \Delta y_k \, \Delta z_k$$

Thm 4 (Fubini's Theorem for Triple Integrals (1st form))

If $f(x,y,z)$ is continuous (in fact, integrable is sufficient) on $B=[a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

Note: Interchanging the order of the coordinates, we also have

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$$

$$= \cdots \text{ in any order of } dx, dy, dz.$$

Def 6 (Triple integral over a general region $D\subset\mathbb{R}^3$)

Let $f(x,y,z)$ be a function on a closed and bounded region $D\subset\mathbb{R}^3$. Then

$$\iiint_D f(x,y,z)\,dV \stackrel{\text{def}}{=} \iiint_B F(x,y,z)\,dV$$

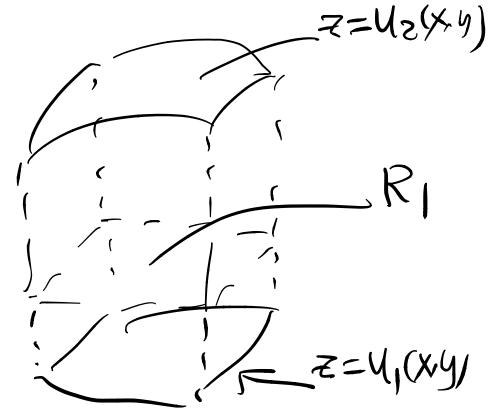
where B is a closed and bounded rectangular box containing D, and

$$F(x,y,z) = \begin{cases} f(x,y,z) & , \text{ if } (x,y,z) \in D \\ 0 & , \text{ if } (x,y,z) \in B \setminus D . \end{cases}$$

<u>Note</u>: As in double integral, this definition is well-defined.

Special types of closed and bounded region $D \subset \mathbb{R}^3$

(1) $D = \{ (x,y,z) : (x,y) \in R_1 ,\ u_1(x,y) \le z \le u_2(x,y) \}$



(2) $D = \left\{ \begin{array}{l} (x,y,z) : (x,z) \in R_2 , \\ v_1(x,z) \le y \le v_2(x,z) \end{array} \right\}$

(3) $D = \left\{ \begin{array}{l} (x,y,z) : (y,z) \in R_3 , \\ w_1(y,z) \le x \le w_2(y,z) \end{array} \right\}$

where R_i, $i = 1,2,3$ are closed and bounded plane region and u_1, u_2 ; v_1, v_2 ; w_1, w_2 are continuous wrt the corresponding variables.

<u>Thm5</u> (<u>Fubini's Thm for triple integrals</u> (strong form))

Let $f(x,y,z)$ be a <u>continuous</u> (integrable) function on D. If D is of type (1) as above, then

$$\iiint_D f(x,y,z)\,dV = \iint_{R_1}\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right]dx\,dy$$

Similarly for types (2) & (3).

<u>Note</u>: Particular, we have (using Fubini's for double integrals): if

$$D = \left\{(x,y,z) : \begin{array}{l} a \le x \le b,\ g_1(x) \le y \le g_2(x) \\ u_1(x,y) \le z \le u_2(x,y) \end{array}\right\}$$

(i.e. R_1 is of type (1)), then

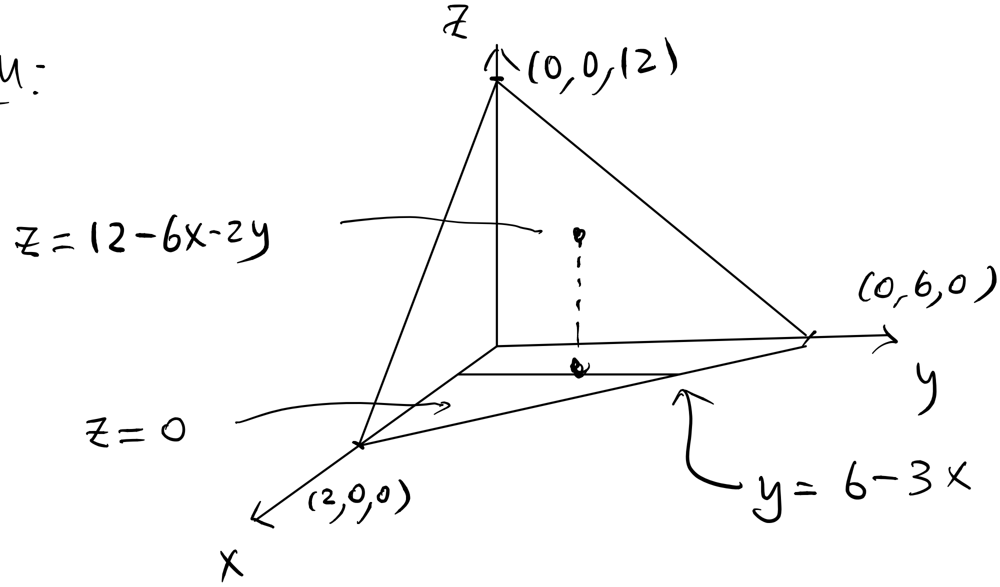
$$\boxed{\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Similarly for other types.

Prop 6 The proprositions 1–4 for double integrals also hold for triple integrals over closed and bounded region in $\mathbb{R}^3$

eg 17 Volume of the bounded region D in the 1^{st} octant enclosed by the plane $6x+2y+z=12$.

Soln:



$\therefore$ D is of special type

$= \{ 0 \leq x \leq 2,\ 0 \leq y \leq 6-3x,\ 0 \leq z \leq 12-6x-2y \}$

$$\Rightarrow \mathrm{Vol}(D) = \iiint_D 1 \, dV$$

$$= \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} dz\, dy\, dx$$

$$= \int_0^2 \int_0^{6-3x} (12-6x-2y)\,dy\,dx \quad \text{(check)}$$

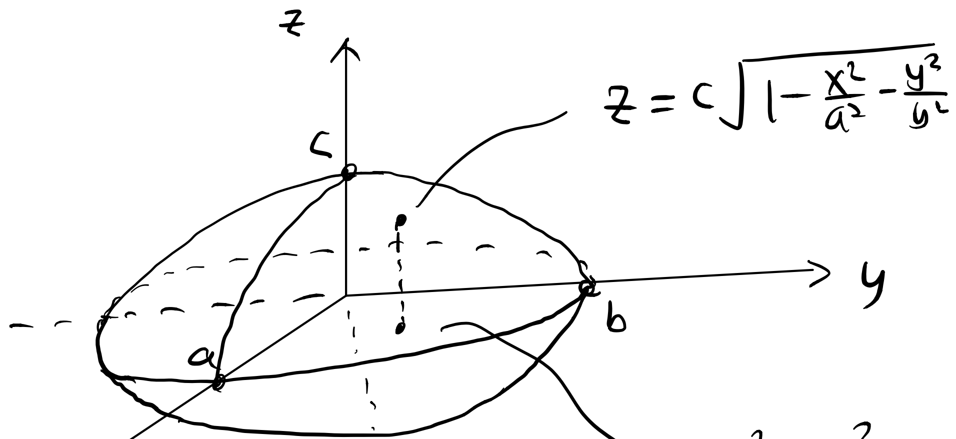
$$= \int_0^2 (9x^2 - 36x + 36)\,dx \quad \text{(check)}$$

$$= 24.$$

(compare eg22 later)

<u>eg18</u> Volume of Ellipsoid

$$D = \left\{ (x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \quad (a,b,c>0)$$



$$R_1 : \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$$

$$= \left\{ 0 \le x \le a,\ 0 \le y \le b\sqrt{1-\frac{x^2}{a^2}} \right\}$$

$$\Rightarrow \mathrm{Vol}(D) = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx.$$

Similarly $$= 8 \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \int_0^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} dx\,dy\,dz.$$

$$= \cdots$$

$$= \frac{4\pi abc}{3} \quad (\text{ exercise } \S 15.5 \text{ problem } 46)$$

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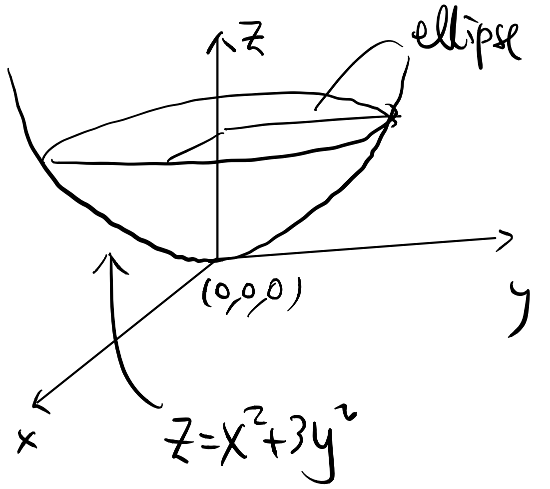
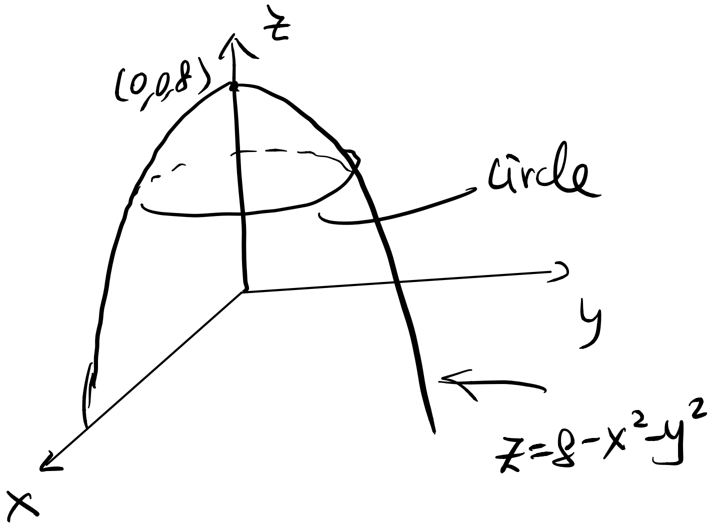
<u>eg 19</u> Find the volume of D enclosed by

$$z = x^2 + 3y^2 \quad \text{and} \quad z = 8 - x^2 - y^2 .$$





At the <u>intersection</u> of the two surfaces

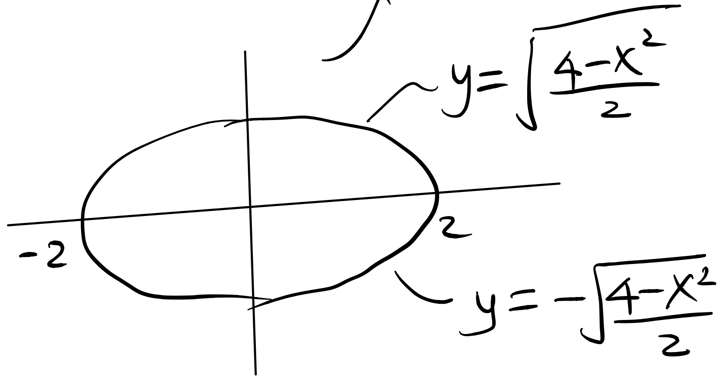
$$x^2+3y^2 = z = 8-x^2-y^2$$

$$\Rightarrow \quad x^2+2y^2=4 \quad \text{ellipse in } xy\text{-plane}$$

$$\left(\text{If } x^2+2y^2 \le 4 \Rightarrow (8-x^2-y^2)-(x^2+3y^2) = 8-2(x^2+2y^2) \ge 0\right)$$

(Note : the intesection cure lies over the ellipse $x^2+2y^2=4$ in the xy-plane.)

$$\Rightarrow D = \{(x,y,z) : x^2+2y^2 \le 4,\ x^2+3y^2 \le z \le 8-x^2-y^2\}$$



$$\Rightarrow D = \left\{ \begin{array}{l} -2 \le x \le 2,\ -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}} \\ x^2+3y^2 \le z \le 8-x^2-y^2 \end{array} \right\}$$

$$\Rightarrow \mathrm{Vol}(D) = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{3}(4-x^2)^{\frac{3}{2}}\, dx \qquad \text{(check)}$$

$$= 8\pi\sqrt{2} \qquad \text{(check)}$$

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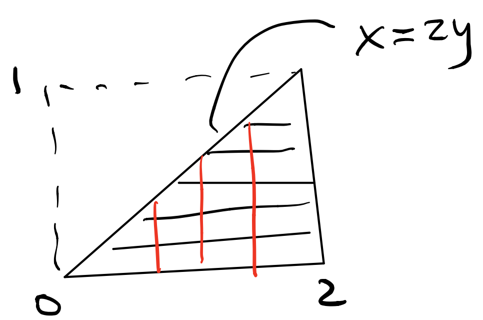
eg 20 Evaluate

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz$$

$$= \int_0^4 \frac{2}{\sqrt{z}} \left(\int_0^1 \int_{2y}^2 \cos x^2 \,dx\,dy \right) dz$$

$$= \left(\int_0^4 \frac{2}{\sqrt{z}} dz \right) \left(\int_0^2 \left(\int_0^{\frac{x}{2}} \cos(x^2)\,dy \right) dx \right)$$



$$= \left(\int_0^4 \frac{2}{\sqrt{z}} dz \right) \left(\int_0^2 (\cos x^2) \left(\int_0^{\frac{x}{2}} dy \right) dx \right)$$

$$= 2\sin 4 \quad (\text{check})$$

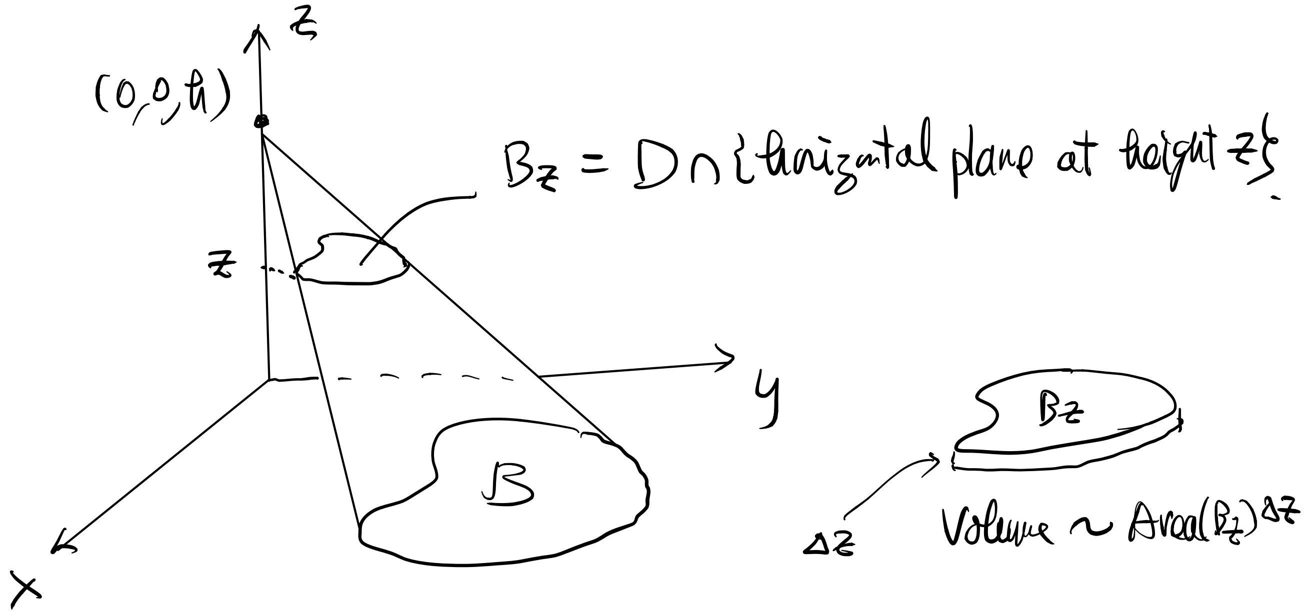
eg 21: Find average value of $F(x,y,z) = xyz$ over the cube $[0,2]^3 \subseteq \mathbb{R}^3$.

Answer: $\quad \text{Average} = \dfrac{\iiint_{[0,2]^3} F(x,y,z)\,dV}{\text{Vol}([0,2]^3)} = 1 \quad (\text{check!})$

$\divideontimes$

eg22 : Let B (base) be a "nice" subset of $\mathbb{R}^2$

Let $D =$ cone in $\mathbb{R}^3$ with base B on xy-plane and vertex $(0,0,h)$



Then $\mathrm{Vol}(D) = \int_0^h \mathrm{Area}(B_z)\,dz$ (By the concept of Riemann sum)

and by similarity :

since the ratio of height : $h-z/h = 1-\frac{z}{h}$,

we have ratio of area : $(1-\frac{z}{h})^2$

$$\Rightarrow \mathrm{Area}(B_z) = (1-\tfrac{z}{h})^2 \mathrm{Area}(B)$$

$$\therefore \mathrm{Vol}(D) = \int_0^h (1-\tfrac{z}{h})^2 \mathrm{Area}(B)\,dz$$

$$= \tfrac{h}{3}\mathrm{Area}(B) \quad \text{※} \quad (\text{Check!})$$