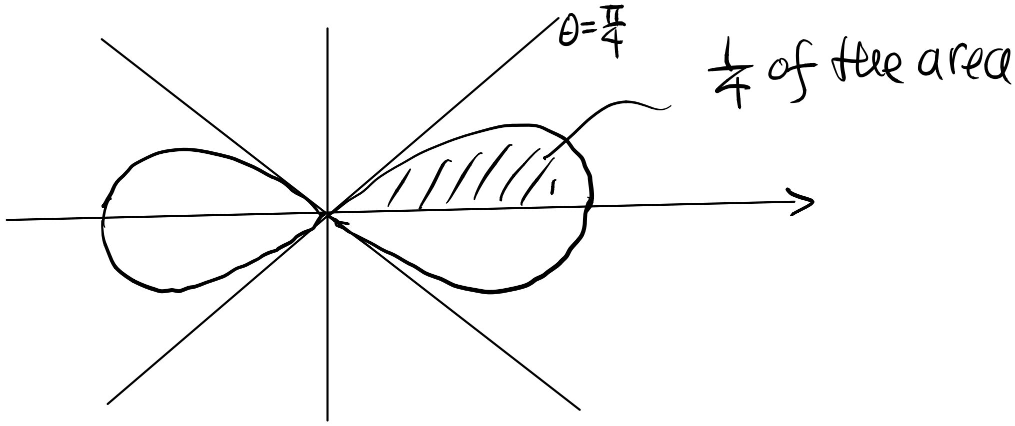
eg14 (cont'd) Find area enclosed by $r^2 = 4\cos 2\theta$

Soln:



(Remark: r is "really" a function of θ, it should be regarded as a "level set": (i) there is no soln. when

$$\frac{\pi}{4} < \theta < \frac{3\pi}{4} \ \&$$

$$\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$$

(ii) in terms of (x,y) coordinates:

$$F(x,y) = (x^2+y^2)^2 - 4(x^2-y^2) = 0$$

which has an critical point at $(0,0)$ on the level set. (Recall: Implicit Function Theorem))

From the symmetry,

$$\text{Area} = 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{4\cos 2\theta}} 1 \cdot r\,dr\,d\theta$$

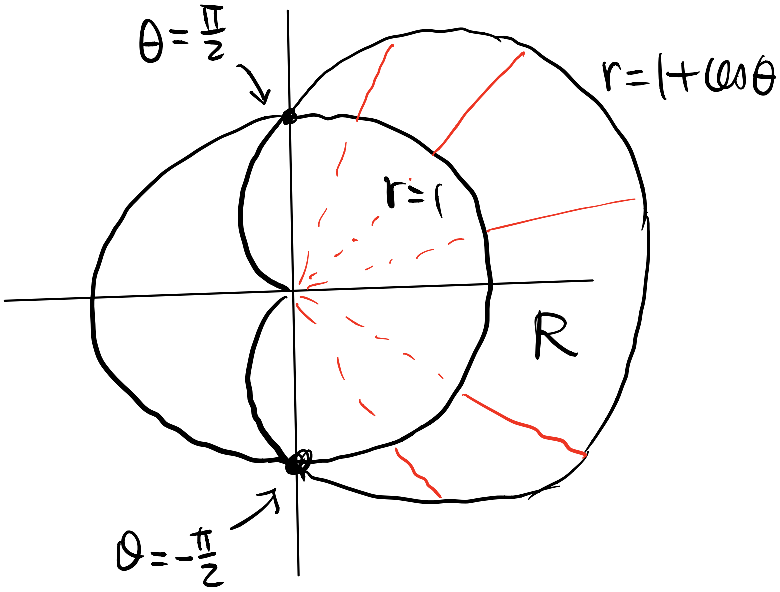
$$= 8\int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta = 4 \quad \text{(check!)}$$

#

eg15: Integrate $f(x,y)=\frac{1}{\sqrt{x^2+y^2}}$ over the region R bounded between

$$\begin{cases} r = 1+\cos\theta & \text{(cardioid)} \\ r = 1 & \text{(circle)} \end{cases}$$

Soln: Intersections: $\cos\theta = 0 \iff \theta = \frac{\pi}{2}+k\pi$



$$\therefore \iint_R f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^{1+\cos\theta} \frac{1}{r}\cdot r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^{1+\cos\theta} dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,d\theta \quad \text{(check!)}$$

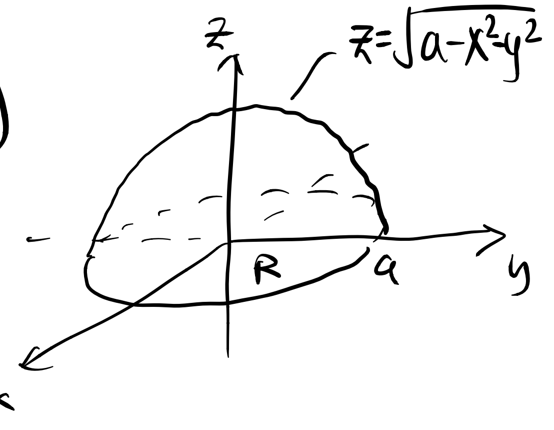
$$= 2 \quad \text{(Check!)}$$

#

eg16 Let $z=\sqrt{a^2-x^2-y^2}$ be a function defined on

$$R=\{(x,y):\ x^2+y^2\leq a^2\}\quad (a>0)$$

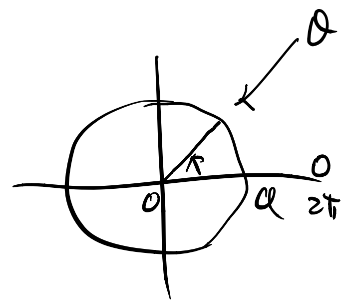
(The graph of z is the hemisphere)



Find the average height of the hemisphere.

Soln :

$$\text{Average height} = \frac{\iint_R z\,dA}{\text{Area}(R)}$$



R=region

$$=\frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\ r\,dr\,d\theta$$

$$=\frac{1}{\pi a^2}\cdot\frac{2\pi a^3}{3}\qquad \text{(check)}$$

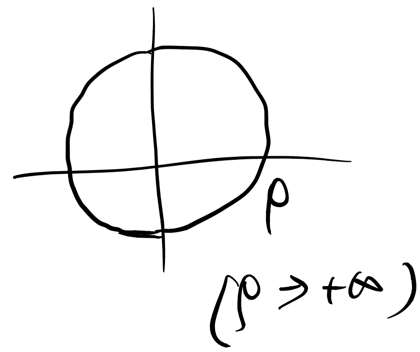
$$=\frac{2a}{3}$$ ✗

eg17 (Improper integral)

Find $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

Soln: $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA$

$$= \lim_{\rho\to+\infty} \iint_{\{x^2+y^2\le\rho^2\}} e^{-(x^2+y^2)}\,dA$$



$$= \lim_{\rho\to+\infty} \int_0^{2\pi}\int_0^{\rho} e^{-r^2}\, r\,dr\,d\theta$$

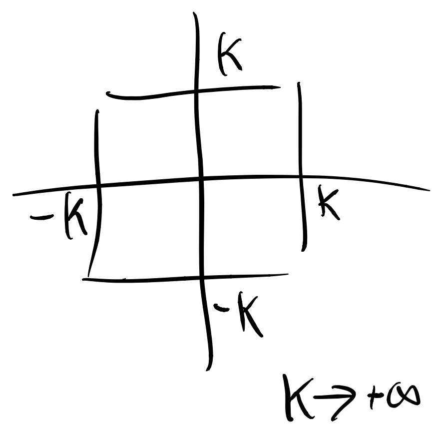
$$= \lim_{\rho\to+\infty} \frac{1}{2}\int_0^{2\pi} (1-e^{-\rho^2})\,d\theta$$

$$= \lim_{\rho\to+\infty} \pi(1-e^{-\rho^2})$$

$$= \pi$$

On the other hand

$$\iint_{R^2} e^{-(x^2+y^2)}\,dA$$



$$= \lim_{K\to+\infty} \int_{-K}^{K}\int_{-K}^{K} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \lim_{K\to+\infty} \int_{-K}^{K} e^{-y^2}\left(\int_{-K}^{K} e^{-x^2}\,dx\right)dy$$

$$= \lim_{k \to +\infty} \left(\int_{-k}^{k} e^{-x^2} dx \right) \left(\int_{-k}^{k} e^{-y^2} dy \right)$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2$$

Hence $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ #

Caution: We are calculating $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$ using two different limiting processes. Why are they equal?

Answer: