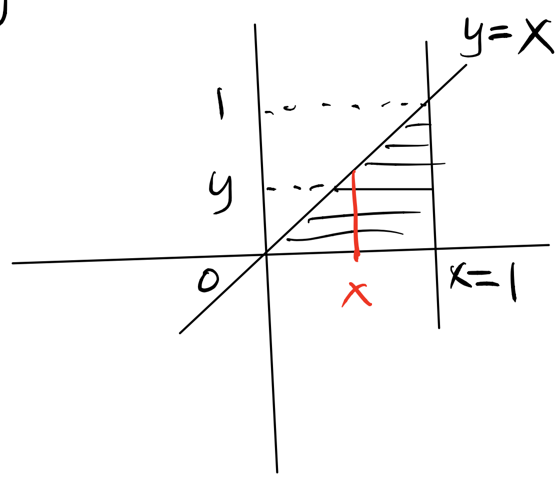
eg 8 : Evaluate $\int_0^1 \left[\int_y^1 \frac{\sin x}{x}\,dx\right] dy$

Soln : Regard $\int_0^1 \left(\int_y^1 \frac{\sin x}{x}\,dx\right) dy$

as a double of $\frac{\sin x}{x}$ over the region

$y \le x \le 1$ and $0 \le y \le 1$



By Fubini's,

$$\int_0^1 \left(\int_y^1 \frac{\sin x}{x}\,dx\right) dy = \int_0^1 \left(\int_0^x \frac{\sin x}{x}\,dy\right) dx$$

$$= \int_0^1 \left(\frac{\sin x}{x} \int_0^x dy\right) dx$$

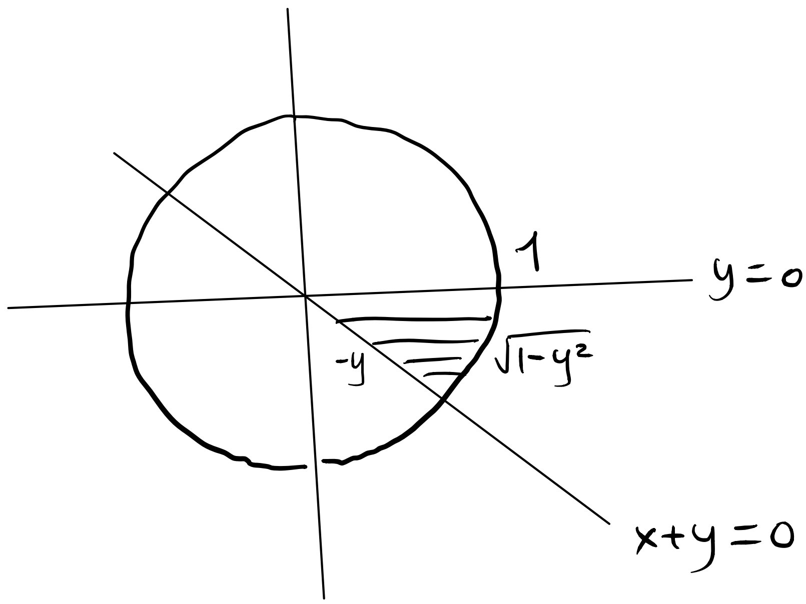
$$= \int_0^1 \sin x\,dx$$

$$= 1 - \cos 1 . \quad ※$$

(Caution: $f(x,y) = \frac{\sin x}{x}$ doesn't define at $x=0$,
why can we use Fubini ? (Ex!))

eg 9 Find $\iint_R x\,dA$, where R is the region in the right half-plane bounded by $y=0$, $x+y=0$, and the unit circle.

Soln: Region R is:



By Fubini's

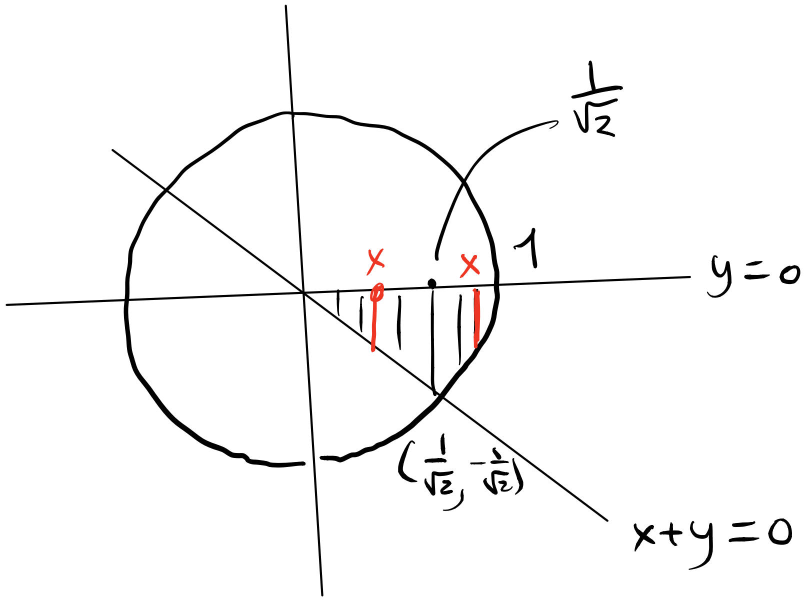
$$\iint_R x\,dA = \int_{-\frac{1}{\sqrt{2}}}^{0}\left(\int_{-y}^{\sqrt{1-y^2}} x\,dx\right)dy$$

(To find the lower limit for integral wrt y, we need to solve $\begin{cases} x^2+y^2=1 \\ x+y=0 \end{cases} \Rightarrow y=-\frac{1}{\sqrt{2}}$ (rejected $+\frac{1}{\sqrt{2}}$))

$$= \int_{-\frac{1}{\sqrt{2}}}^{0}\left(\frac{1}{2}-y^2\right)dy \qquad \text{(check!)}$$

$$= \frac{1}{3\sqrt{2}}$$ (check!)

Alternatively:



$$\iint_R x\,dA$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \left(\int_{-x}^{0} x\,dy \right) dx$$

$$+ \int_{\frac{1}{\sqrt{2}}}^{1} \left(\int_{-\sqrt{1-x^2}}^{0} x\,dy \right) dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x^2\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} x\sqrt{1-x^2}\,dx$$ (check)

$$= \frac{1}{3\sqrt{2}}$$ (check!)

(More complicated in expressing the integral, but easier to do the 1st integration.)

Applications

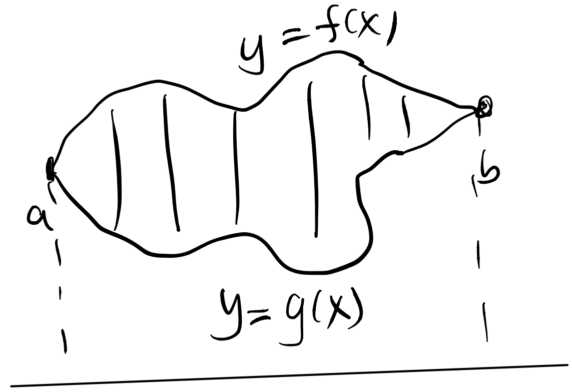
(1) Area (of (good) region $R \subset \mathbb{R}^2$)

Def 3:
$$\text{Area}(R) = \iint_R 1 \, dA$$

Then Fubini's Thm implies the well-known formula

$$\text{Area}(R) = \int_a^b [f(x) - g(x)] \, dx$$ if R is the region bounded by the curves $y = f(x) \geq g(x) = y$ for $a \leq x \leq b$.

$(f(a) = g(a)$ & $f(b) = g(b))$

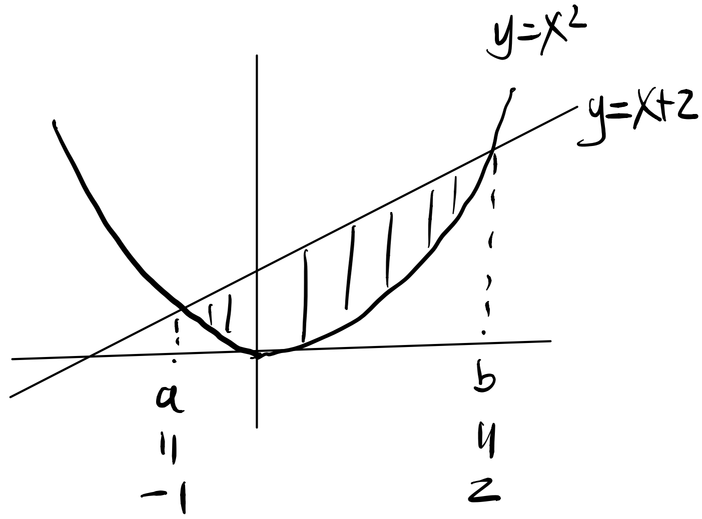


eg 10: Area bounded by $y = x^2$ and $y = x + 2$

Soln: Solving $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$

$$\Rightarrow \quad x^2 = x + 2$$

$\Rightarrow \quad x = -1, 2$



Then (by Fubini's)

$$\text{Area} = \int_{-1}^{2}\int_{x^2}^{x+2} 1 \, dy \, dx$$

$$= \int_{-1}^{2} (x+2-x^2) \, dx = \frac{9}{2} \quad (\text{check!})$$

(2) Average (of a function over a region)

Let $f : R \xrightarrow{} \mathbb{R}$ ($R \subset \mathbb{R}^2$) be an integrable function.

> **Def 4**: The average value of f over R
>
> $$= \frac{1}{\text{Area}(R)} \iint_R f(x,y) \, dA$$

eg 11: Let $f(x,y) = x\cos(xy)$, $R = [0,\pi] \times [0,1]$.

Find average of f over R.

Soln: Average of f over $R = \dfrac{1}{\text{Area}(R)} \displaystyle\iint_R f(x,y) \, dA$

$$= \frac{1}{\pi} \int_0^{\pi} \left(\int_0^1 x \cos(xy)\, dy \right) dx$$

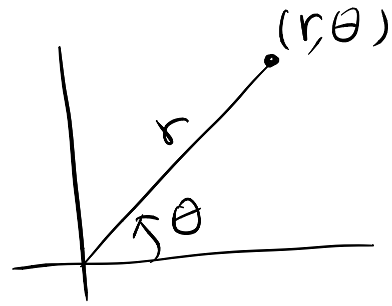
$$= \frac{1}{\pi} \int_0^{\pi} \sin x \, dx \qquad \text{(check!)}$$

$$= \frac{2}{\pi} \qquad \text{(check!)} \quad \#$$

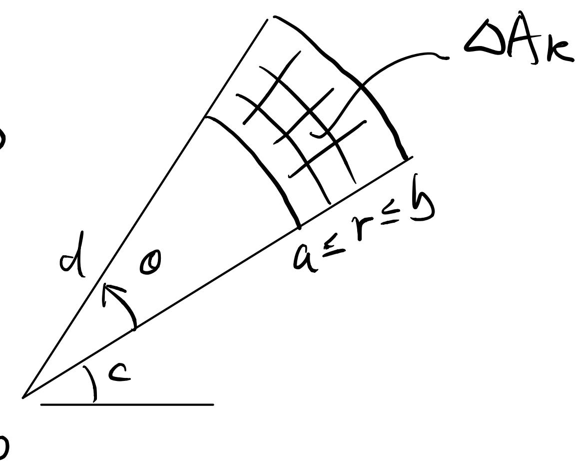
Double integral in polar coordinates

$$(r, \theta) \longleftrightarrow (x, y)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



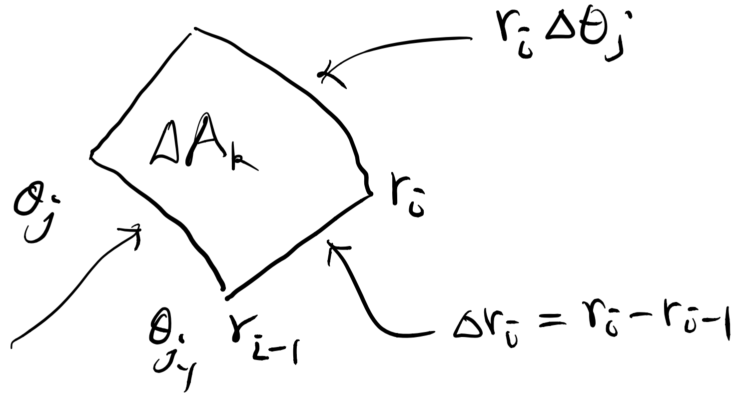
$$\begin{cases} a \le r \le b \\ c \le \theta \le d \end{cases} \longrightarrow$$



Idea:

$$\sum_k f(\text{point}_k)\, \Delta A_k$$

what is ΔA_k (approximately)



$r_{i-1}\Delta\theta_j$
$= r_{i-1}(\theta_j - \theta_{j-1})$

$$\Delta A_k \simeq (r_i \Delta\theta_j)\Delta r_i \simeq (r_{i-1}\Delta\theta_j)\Delta r_i$$

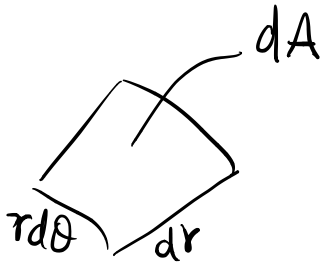
Hence $\Delta x \Delta y \simeq r \Delta\theta \, \Delta r$

So $\displaystyle\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$

$$= \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

<u>method to remember the famula</u>

$$dA = dx\,dy = r\,dr\,d\theta$$



Double integral of f over $R = \left\{ (r,\theta) : \begin{array}{l} a \le r \le b \\ c \le \theta \le d \end{array} \right\}$

in polar coordinates is

$$\iint_R f(r,\theta)\, r\,dr\,d\theta = \int_c^d \left[\int_a^b f(r,\theta)\, r\,dr \right] d\theta$$

$$= \int_a^b \left[\int_c^d f(r,\theta)\,d\theta \right] r\,dr$$

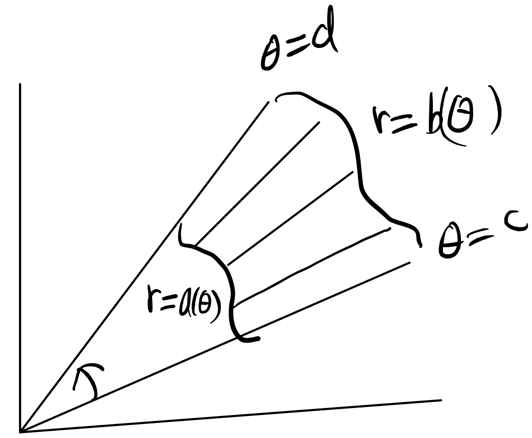
Remark: This is a special case of the change of variable formula. The "extra" factor "r" in the integrand is in fact

$$r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

the Jacobian determinant of the change of coordinates.

More generally:

Thm 3 If R is a (closed and bounded) region with $c \le \theta \le d$ and $a(\theta) \le r \le b(\theta)$.



And $f: R \to \mathbb{R}$ is a continuous function on R

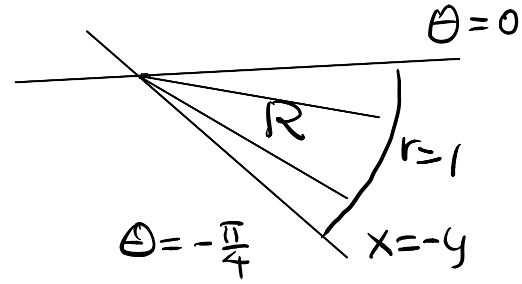
Then $\iint_R f(x,y)\,dA$

$$= \int_c^d \left[\int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr \right] d\theta.$$

(remember this extra "r")

eg 12: Back to our previous example 9,

$f(x,y) = x = r\cos\theta$



and

$$\int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-y}^{\sqrt{1-y^2}} x\,dx\,dy$$

$$= \int_{-\frac{\pi}{4}}^{0} \left[\int_0^1 r\cos\theta \cdot r\,dr \right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{0} \left(\int_0^1 r^2\,dr \right) \cos\theta\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{0} \frac{1}{3}\cos\theta\,d\theta \qquad \text{(check!)}$$

$$= \frac{1}{3\sqrt{2}} \qquad \text{(check!)}$$

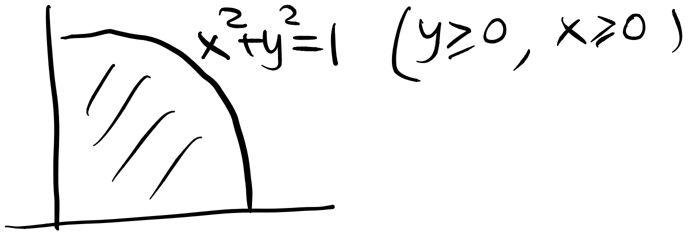
#

<u>eg13</u> Convert integrals between Cartesian and Polar coordinates.

(a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta\cos\theta\,dr\,d\theta$

(b) $\int_1^2 \int_0^{\sqrt{2x-x^2}} y\,dy\,dx$

<u>Soln</u>: (a) Region: $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$



$$\Rightarrow \int_0^{\frac{\pi}{2}}\int_1^1 r^3 \sin\theta\cos\theta \, dr\, d\theta$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} (r\sin\theta)(r\cos\theta)\, \underbrace{r\,dr\,d\theta}$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} xy\, dy\, dx$$

don't faget this r.

(b) $\displaystyle\int_1^2\int_0^{\sqrt{2x-x^2}} y\, dy\, dx$

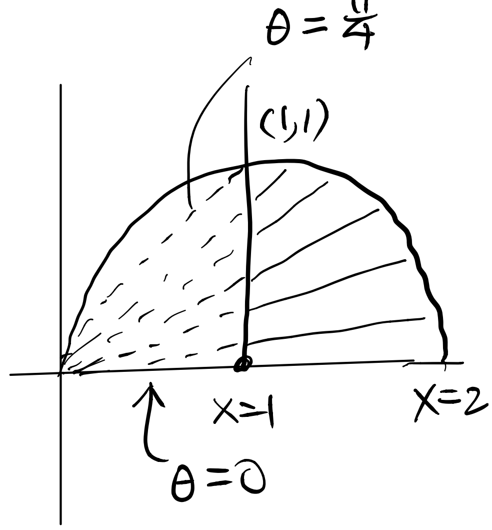
The regim is $1\leq x\leq 2$, $0\leq y\leq \sqrt{2x-x^2}$

upper limit in y is given by the curve

$$y=\sqrt{2x-x^2}$$

$$\Leftrightarrow \quad y^2 = 2x-x^2$$

$$\Leftrightarrow \quad (x-1)^2+y^2=1$$



The curve $x=1$

$$\Leftrightarrow \quad r\cos\theta = 1$$

$$\Leftrightarrow \quad r = \frac{1}{\cos\theta} = \sec\theta \quad (0 \le \theta \le \tfrac{\pi}{4})$$

The curve $(x-1)^2 + y^2 = 1$

$$\Leftrightarrow \quad (r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1$$

$$\Leftrightarrow \quad r^2 - 2r\cos\theta = 0$$

$$\Leftrightarrow \quad r - 2\cos\theta = 0 \quad (\text{since } r > 0)$$

$$\Leftrightarrow \quad r = 2\cos\theta \quad (0 \le \theta \le \tfrac{\pi}{4})$$

Hence

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} y\,dy\,dx$$

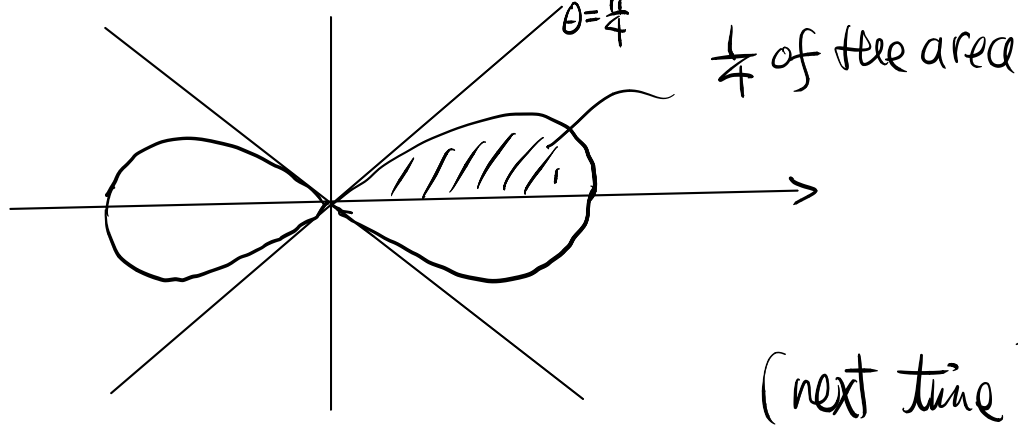
remember this extra r!

$$= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} r\sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} r^2 \sin\theta \, dr\, d\theta \quad \text{※}$$

eg 14: Find area enclosed by $r^2 = 4\cos 2\theta$

Soln:



(next time).