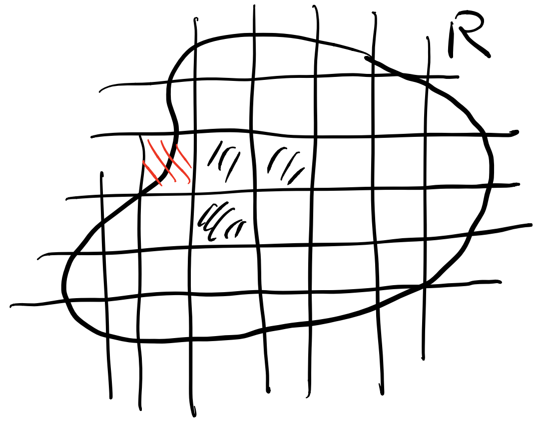
Double integrals over General Regions

For non-rectangular bounded (closed) region R,
one can define similarly
the concept of "Riemann sum"



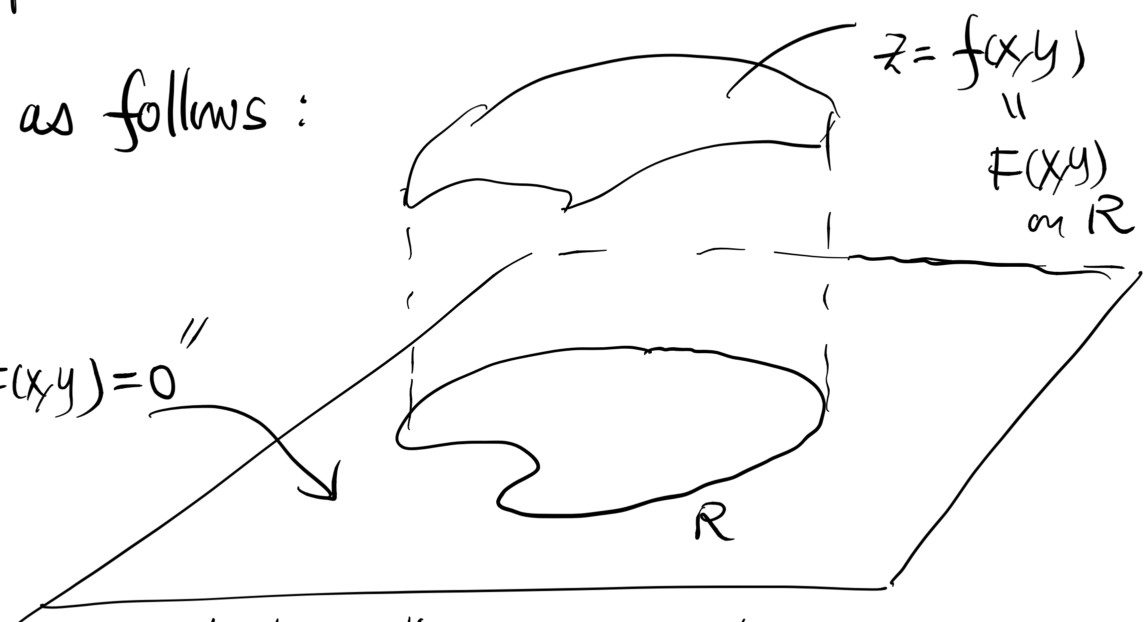
There are two ways to form
a "Riemann sum":

(i) sum over all subrectangles completely inside R

(ii) sum over all "subrectangles" with non-empty intersection with R.

(More complicate than 1-variable)

Or define as follows:



R' = "large" rectangle containing R

Def 2: Let R be a bounded region and $f(x,y)$ be a function defined on R. For any rectangle $R' \supset R$, define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in R \\ 0 & \text{if } (x,y) \in R' \setminus R \end{cases}$$

Then the integral of f over R is defined by

$$\iint_R f(x,y)\, dA = \iint_{R'} F(x,y)\, dA .$$

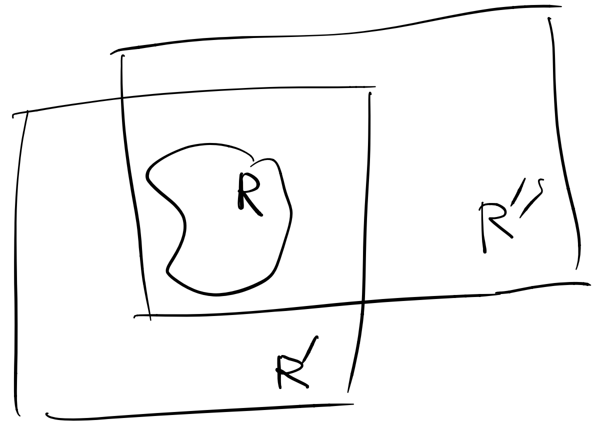
Remark: The definition is well-defined:

if R'' is another rectangle s.t. $R'' \supset R$ and

$$\tilde{F}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in R \\ 0 & \text{if } (x,y) \in R'' \setminus R \end{cases}$$

Then

$$\iint_{R''} \tilde{F}(x,y)\, dA = \iint_{R'} F(x,y)\, dA$$



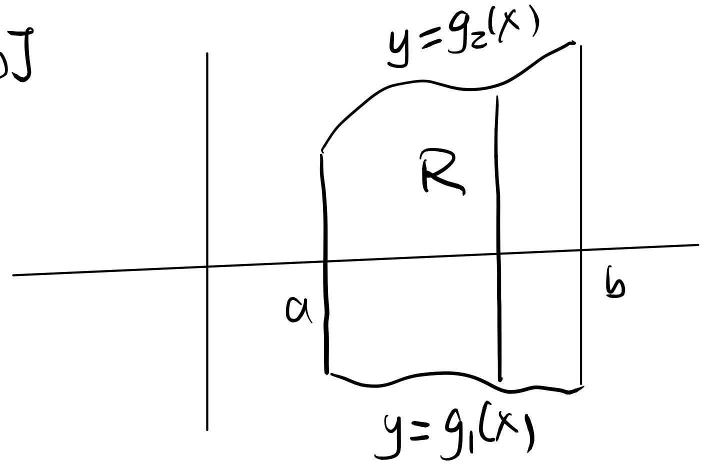
(by Riemann sum argument.)

Prop 5: The propositions 1–4 hold if we replace "closed rectangle" by "closed and bounded region".

Important special types of bounded regions R

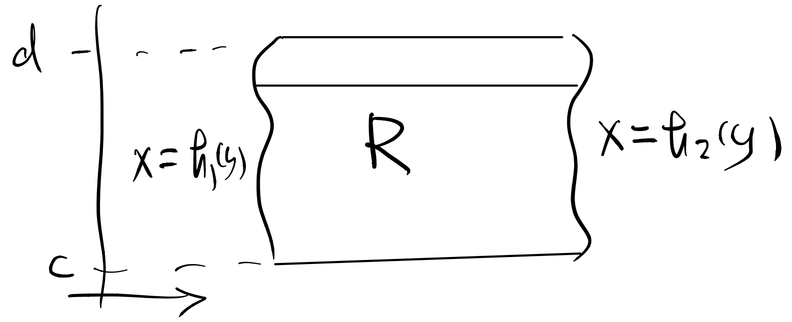
(1) $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

where g_1 and g_2 are "continuous" functions on $[a,b]$



(2) $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

where h_1 and h_2 are "continuous" functions on $[c,d]$



For these 2 types of bounded regions, we have

Thm 2 (Fubini's Theorem (Stronger version))

Let $f(x,y)$ be a continuous function on a closed and bounded region R.

(1) If R is of type (1) as above, then

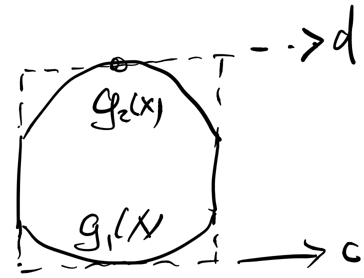
$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

(2) If R is of type (2) as above, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right] dy$$

Pf: Type (1) Extend $f(x,y)$ to $F(x,y)$ as above on a rectangle $R' = [a,b]\times[c,d]$ such that

$c = \min_{[a,b]} g_1(x)$ and $d = \max_{[a,b]} g_2(x)$.



By definition 2,

$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

$$\underset{(1^{st}\text{ form})}{\overset{\text{Fubini}}{=}} \int_a^b \left[\int_c^d F(x,y)\,dy \right] dx$$

(f continuous on $R \Rightarrow F$ continuous on R' except possibly along the boundary curves on F. Hence by remark (ii) of Prop 2, F is integrable over R'. And the Fubini theorem (1^{st} form) is in fact true for integrable functions on a rectangle.)

Now note that $F(x,y) = 0$ for $y < g_1(x)$ or $y > g_2(x)$

and $F(x,y) = f(x,y)$ for $g_1(x) \leq y \leq g_2(x)$

$$\therefore \iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx .$$

Type (2) can be proved similarly.

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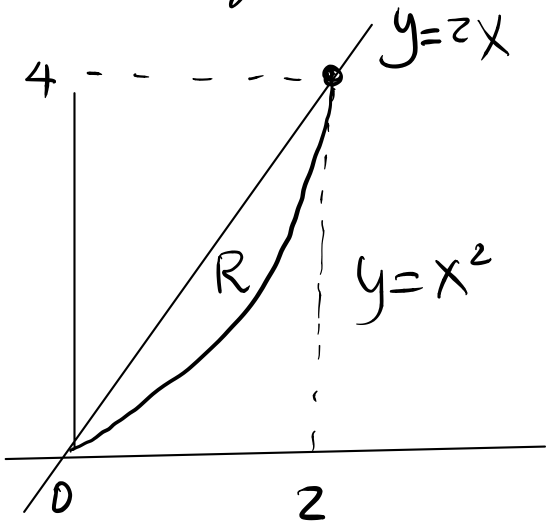
eg7 Integrate $f(x,y) = 4y+2$ over the region bounded by $y=x^2$ and $y=2x$.

Soln:



By Fubini's $\displaystyle\iint_R f(x,y)\,dA$

$$= \int_0^2 \left[\int_{x^2}^{2x} f(x,y)\,dy \right] dx$$ (regarding R as type (I))

$$= \int_0^2 \left[\int_{x^2}^{2x} (4y+2)\,dy \right] dx$$

$$= \int_0^2 (-2x^4+6x^2+4x)\,dx$$ (check!)

$$= \frac{56}{5}$$ (check!)

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In fact, R is also type (2) and Fubini's

$$\Rightarrow \iint_R f(x,y)\,dA = \int_0^4 \left[\int_{\frac{y}{2}}^{\sqrt{y}} (4y+2)\,dx \right] dy$$

$$= \int_0^4 \left[(4y+2) \int_{\frac{y}{2}}^{\sqrt{y}} dx \right] dy$$

$$= \cdots = \frac{56}{5} \quad (\text{check!})$$