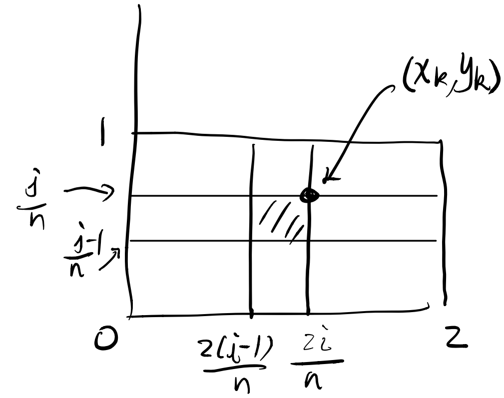
eg 2 (cont'd)

Using the uniform partitions



$P_1 = \{0, \frac{2}{n}, \frac{4}{n}, \cdots, 2\}$ of $[0,2]$

$P_2 = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$ of $[0,1]$

$\begin{pmatrix} i = 1, \cdots, n \\ j = 1, \cdots, n \end{pmatrix}$

$\Rightarrow$ a particular subrectangle is

$$R_k = \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right].$$

One may choose the point $(x_k, y_k) = \left(\frac{2i}{n}, \frac{j}{n}\right)$

and consider the Riemann sum

$$\sum_k f(x_k, y_k) \Delta A_k$$

$$= \sum_k (x_k \cdot y_k^2)\left(\frac{2}{n} \cdot \frac{1}{n}\right)$$

$$= \sum_k \left(\frac{2i}{n}\right)\left(\frac{j}{n}\right)^2 \cdot \frac{2}{n^2}$$

($\sum_k$ means "sum over all subrectangles") i.e. sum over all i, j

$$= \sum_{i,j=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{j}{n}\right)^2 \cdot \frac{2}{n^2}$$

$$= \frac{4}{n^5} \sum_{i,j=1}^{n} i j^2$$

$$= \frac{4}{n^5} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} i j^2 \right) = \frac{4}{n^5} \sum_{i=1}^{n} \left(i \sum_{j=1}^{n} j^2 \right)$$

$$= \frac{4}{n^5} \sum_{i=1}^{n} \left(i \cdot \frac{n(n+1)(2n+1)}{6} \right)$$

$$= \frac{4}{n^5} \cdot \frac{n(n+1)(2n+1)}{6} \sum_{i=1}^{n} i$$

$$= \frac{4}{n^5} \frac{n(n+1)(2n+1)}{6} \cdot \frac{n(n+1)}{2}$$

$$\to \frac{4 \cdot 2}{6 \cdot 2} = \frac{2}{3} \quad \text{as } n \to \infty$$

$$\therefore \iint_{[0,2]\times[0,1]} (xy^2)\,dx\,dy = \frac{2}{3} \quad \#$$

Very tedious calculation!

Hence we need the following Theorem:

Thm 1 (Fubini's Theorem (1st form))

If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$,

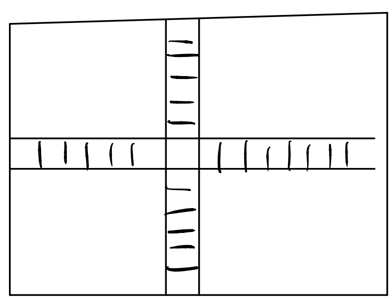
then $$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

$$= \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

The last 2 Integrals above are called iterated integrals.

Ideas:

sum horizontally first gives $\int_c^d \left[\int_a^b f(x,y)dx\right]dy$



sum vertically first gives $\int_a^b \left[\int_c^d f(x,y)dy\right]dx$

(Pf: Omitted)

eg 3: Using Fubini to calculate $\iint_R xy^2\,dx\,dy$,

where $R=[0,2]\times[0,1]$.

Soln: By Fubini,

$$\iint_R xy^2\,dx\,dy = \int_0^2 \left(\int_0^1 xy^2\,dy \right) dx$$

$$= \int_0^2 \left[\frac{xy^3}{3} \right]_{y=0}^{y=1} dx$$

$$= \int_0^2 \frac{x}{3}\,dx = \frac{2}{3}$$

or $$\iint_R xy^2\,dx\,dy = \int_0^1 \left(\int_0^2 xy^2\,dx \right) dy$$

$$= \int_0^1 \left[\frac{x^2}{2} y^2 \right]_{x=0}^{x=2} dy$$

$$= \int_0^1 2y^2\,dy = \frac{2}{3}$$

Much easier than using Riemann sum! ✱

eg4: Some times the "order" of the iterated integrals is impatant in practical calculations!

Find $\displaystyle\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA$

Soln: $\displaystyle\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA$

$$= \int_0^{\pi} \left[\int_0^1 x \sin(xy)\, dx \right] dy$$

Integration-by-parts give

$$= \int_0^{\pi} \left[-\frac{\cos y}{y} + \frac{\sin y}{y^2} \right] dy$$

not easy to integrate !

On the other order :

$$\iint_{[0,1]\times[0,\pi]} x \sin(xy)\, dA = \int_0^1 \left[\int_0^{\pi} x \sin(xy)\, dy \right] dx$$

$$= \int_0^1 \left[-\cos xy \right]_{y=0}^{y=\pi} dx$$

$$= \int_0^1 \left[-\cos \pi x + 1 \right] dx$$

$$= \left[-\frac{1}{\pi} \sin \pi x + x \right]_0^1$$

$$= 1 \qquad (\text{Easy!})$$

<u>Caution</u> : <u>Not</u> all function integrable over a (closed) rectangle.

eg5 Let $R = [0,1] \times [0,1]$,

$$f(x,y) = \begin{cases} 0, & \text{if both } x \text{ \& } y \text{ are rational} \\ 1, & \text{otherwise} \end{cases}$$

Then f is not integrable over R.

Soln: $\forall$ subdivision (partition) P of R

$= R_1 \cup R_2 \cdots \cup R_n$,

one can find point $(x_k, y_k) \in R_k$ s.t.

x_k, y_k are both rational ($\forall k = 1, \cdots, n$)

The corresponding Riemann sum equals

$$\sum_{k=1}^{n} f(x_k, y_k) \Delta A_k = \sum_{k=1}^{n} 0 \cdot \Delta A_k = 0 \to 0 \text{ as } \|P\| \to 0$$

On the other hand, we can also find $(\hat{x}_k, \hat{y}_k) \in R_k$ s.t. at least one of the $\hat{x}_k, \hat{y}_k$ is irrational.

The corresponding Riemann sum equals

$$\sum_{k=1}^{n} f(\hat{x}_k, \hat{y}_k) \Delta A_k = \sum_{k=1}^{n} 1 \cdot \Delta A_k = \text{Area}(R) = 1$$

$\to 1$

as $\|P\| \to 0$

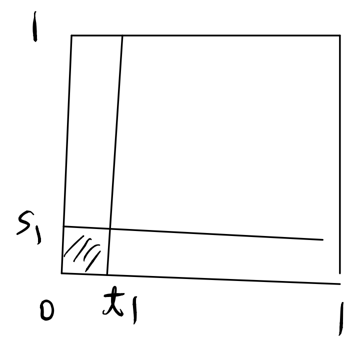
So f cannot be integrable over R. #

eg 6: Let $R=[0,1]\times[0,1]$

$$f(x,y)=\begin{cases}\frac{1}{xy} & \text{if } x\neq 0 \ \& \ y\neq 0\\ 0 & \text{if } x=0 \text{ or } y=0\end{cases}$$

Then f is not integrable over R.

Pf: In any partition P of R, there is a sub-rectangle



$R_1=[0,t_1]\times[0,s_1]$.

we can take

$$(x_1,y_1)=(t_1^2,s_1^2)\in R_1=[0,t_1]\times[0,s_1]$$

$$\left(\text{since } 0<t_1^2<t_1<1,\ 0<s_1^2<s_1<1\right)$$

Then Riemann sum

$$\sum_{k=1}^{n} f(x_k,y_k)\Delta A_k \geq f(x_1,y_1)\Delta A_1 \quad (\text{since } f\geq 0)$$

$$= \frac{1}{t_1^2 s_1^2} \cdot (t_1 \cdot s_1)$$

$$= \frac{1}{t_1 s_1}$$

Since $0 < t_1, s_1 \le \|P\|$,

$t_1, s_1 \to 0$ as $\|P\| \to 0$

$$\Rightarrow \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k \ge \frac{1}{t_1 s_1} \to \infty \text{ as } \|P\| \to \infty$$

Limit doesn't exist and f is not integrable over R.

$\#$

By egs 5 & 6, we see that "condition" is needed to ensure integrability of a function over (closed) rectangle.

Prop 1 Let $R = [a,b] \times [c,d]$ be a closed rectangle, and $f(x,y)$ be an integrable function over R, then f is bounded on R

(i.e. $\exists M > 0$ s.t. "$|f(x,y)| \le M$, $\forall (x,y) \in R$")

Pf: Omitted (See eg 6 above for an idea of proof)

Prop 2: Let $R = [a,b] \times [c,d]$ be a closed rectangle, and $f(x,y)$ be a continuous function on R, then f is integrable over R.

Pf: Omitted (see proof in 1-variable case in MATH2060 for an ideal of proof.)

Remarks (i) Note that a continuous function on closed rectangle is always bounded (Props 1 & 2 are consistent.)

(ii) Prop 2 can be generalized to a bounded function on a closed rectangle with a "small" set of discontinuity. The precise concept is "measure zero set" (see MATH4050 Real Analysis).

For us, we have:

For function over closed rectangle

(a) bounded + "continuous except finitely many points" $\Rightarrow$ integrable.

(b) bounded + "continuous except finitely many differentiable curves" $\Longrightarrow$ integrable.

Furthermore, we have

Prop3: Let $R = [a,b] \times [c,d]$ be a closed rectangle, $f(x,y)$ and $g(x,y)$ be functions on R, and $k \in \mathbb{R}$ is a constant.

(1) If f & g are <u>integrable</u> over R, then $f \pm g$ and kf are <u>integrable</u> over R

(2) In case of (1), we have

$$\iint_R [f \pm g](x,y)\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$$

and

$$\iint_R k f(x,y)\, dA = k \iint_R f(x,y)\, dA$$

Pf: Omitted (Obvious from the concept of Riemann sum)

<u>Remark</u>: This Prop3 implies that the set of integrable functions over R form a "vector space over $\mathbb{R}$", and "(double) integral" is <u>linear</u> (when the rectangle R is fixed)

Prop 4 (a) If $f(x,y) \geq 0$ is an integrable function on a closed rectangle R, then

$$\iint_R f(x,y)\,dA \geq 0$$

(b) If R_1 and R_2 be two closed rectangles such that $\operatorname{int} R_1 \cap \operatorname{int} R_2 = \emptyset$, then

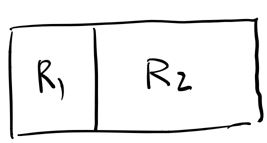
$$\iint_{R_1 \cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

for integrable function f over $R_1 \cup R_2$

Pf: Omitted (Obvious from the concept of Riemann sum)

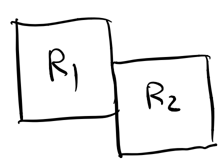
Note: Various situation for $\operatorname{int} R_1 \cap \operatorname{int} R_2 = \emptyset$:

(1)

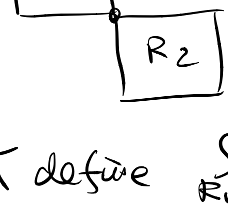


$R_1 \cap R_2 =$ common edge

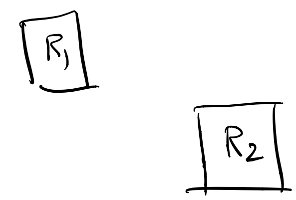
(2)



(3)



(4)



We haven't define $\iint_{R_1 \cup R_2} f(x,y)\,dA$ for cases (2)-(4).

Double Integrals over General Regions