Double Integrals

Recall: In one-variable, "integral" is regarded as "limit" of "Riemann sum" (take MATH2060 for rigorous treatment)

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

where
$$\begin{cases} f \text{ is a function on the interval } [a,b] \\ P \text{ is a } \underline{\text{partition}} \quad a=t_0<t_1<\cdots<t_n=b \\ x_k\in[t_{k-1},t_k] \text{ and } \Delta x_k = t_k - t_{k-1} \\ \|P\| = \max_k |\Delta x_k| \end{cases}$$

Remark: We usually use uniform partition P:

$$a=t_0<t_1=a+\frac{1}{n}(b-a)<t_2<\cdots$$
$$\cdots<t_k=a+\frac{k}{n}(b-a)<\cdots<t_n=b$$

Then $\|P\|=\frac{1}{n}$ $(\to 0, \text{ as } n\to\infty)$

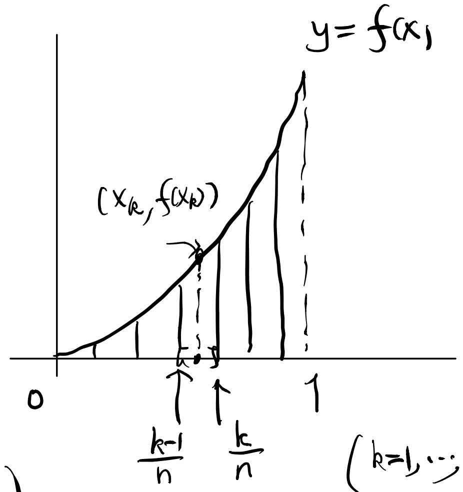
eg1: Find $\int_0^1 x^2 dx$ (i.e. $f(x)=x^2$ on $[0,1]$)

Soln: (1) One may choose $x_k=\frac{k-1}{n}\in\left[\frac{k-1}{n},\frac{k}{n}\right]$

Then

$$S_n = \sum_{k=1}^{n} x_k^2 \, \Delta x_k$$

$$= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \cdot \frac{1}{n}$$



$$= \frac{1}{n^3} \cdot \frac{(n-1)\cdot n \cdot (2n-1)}{6} \quad \text{(check!)}$$

$$= \frac{1}{6}\left(1-\frac{1}{n}\right)\cdot 1 \cdot \left(2-\frac{1}{n}\right)$$

(uniform partition $\|P\| = \frac{1}{n} = \Delta x_k$, $k=1,\cdots,n$)

$$\to \frac{1}{3} \quad \text{as } n \to \infty \ (\|P\| \to 0)$$

$$\therefore \int_0^1 x^2 \, dx = \frac{1}{3}$$

(2) Or we may choose $x_k = \frac{k}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$

(Will we get different answer ?)

Then

$$S_n = \sum_{k=1}^{n} x_k^2 \, \Delta x_k = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n}$$

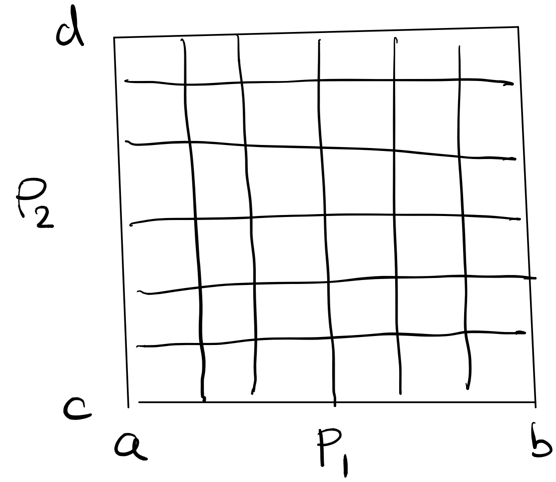
$$= \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \to \frac{1}{3} \quad \text{as } n \to \infty \ (\text{i.e. } \|P\| \to 0)$$

Remark: We can use any $x_k \in [t_{k-1}, t_k]$, and still get the same $\int_0^1 x^2 \, dx = \frac{1}{3}$. #

This concept can be generalised to any dimension.

For 2-dim., let we first consider a function $f(x,y)$ defined on a rectangle

$$R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$



Then we can subdivide R into sub-rectangles by using partitions P_1 of $[a,b]$ and

P_2 of $[c,d]$.

Denote $P = P_1 \times P_2$ (partition (subdivision) of R)

and $\|P\| = \max(\|P_1\|, \|P_2\|)$

Let the sub-rectangles be R_k, $k = 1, \dots, n$

with areas ΔA_k,

Choose point $(x_k, y_k) \in R_k$, then consider "Riemann sum"

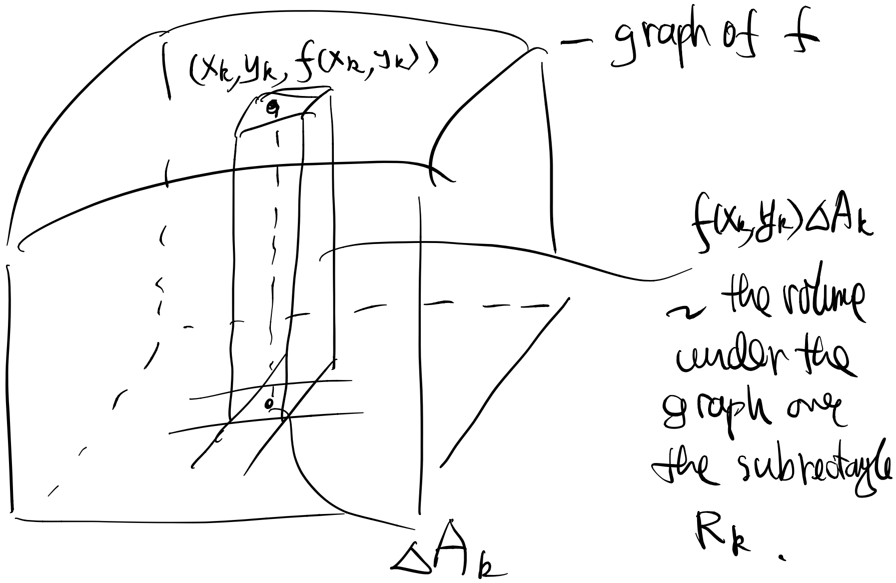
$$S(f;P) = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

<u>Def 1</u>: The function f is said to be <u>integrable</u> over R if $\lim_{\|P\|\to 0} S(f;P) = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$ exists & independent of the choose of $(x_k, y_k) \in R_k$.

In this case, the limit is called the (<u>double</u>) <u>integral of f over R</u> and is denoted by

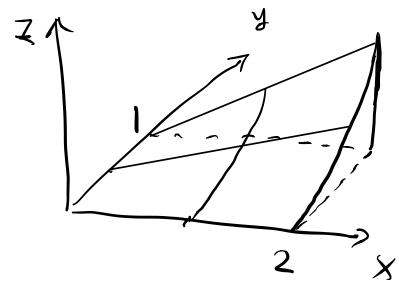
$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

<u>Remark</u>: Same as 1-variable, the <u>double integral</u> of f $(f \geq 0)$ over R can be interpreted as <u>volume</u> <u>under the graph of f</u>:



eg2 $R = [0,2] \times [0,1]$, $f(x,y) = xy^2$

Find $\iint_R xy^2 \, dx\, dy$.



Soln:

Next time: we will use

uniform partition $P_1 = \{0, \frac{2}{n}, \frac{4}{n}, \cdots 2\}$ of $[0,2]$

$P_2 = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$ of $[0,1]$

to calculate the integral.