Unified treatment of Green's, Stokes', and Divergence Theorems

Stokes' Thm in notations of differential forms (in $\mathbb{R}^3$)

Working definition of differential forms

(1) A differential 1-form (or simply 1-form)

is a linear combination of the symbols dx, dy & dz:

$$\boxed{\omega = \omega_1\, dx + \omega_2\, dy + \omega_3\, dz}$$

with coefficients $\omega_1, \omega_2, \omega_3$ are functions on $\mathbb{R}^3$.

eg: The total differential of a smooth function f

is a differential 1-form:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

(2) Wedge product: Let "$\wedge$" be an operation such that

$$\begin{cases} dx \wedge dx = dy \wedge dy = dz \wedge dz = 0 \\ dx \wedge dy = -dy \wedge dx \\ dy \wedge dz = -dz \wedge dy \\ dz \wedge dx = -dx \wedge dz \end{cases}$$

and satisfies other usual rules in arithmetic.

i.e. If $\omega = \omega_1\, dx + \omega_2\, dy + \omega_3\, dz$

$\eta = \eta_1\, dx + \eta_2\, dy + \eta_3\, dz$

Then we have

$$\omega\wedge\eta = (\omega_1 dx + \omega_2 dy + \omega_3 dz)\wedge(\eta_1 dx + \eta_2 dy + \eta_3 dz)$$

$$\begin{aligned}
&= \omega_1 \cancelto{0}{dx\wedge\eta_1 dx} + \omega_2 dy\wedge\eta_1 dx + \omega_3 dz\wedge\eta_1 dx \\
&\quad + \omega_1 dx\wedge\eta_2 dy + \omega_2 \cancelto{0}{dy\wedge\eta_2 dy} + \omega_3 dz\wedge\eta_2 dy \\
&\quad + \omega_1 dx\wedge\eta_3 dz + \omega_2 dy\wedge\eta_3 dz + \omega_3 \cancelto{0}{dz\wedge\eta_3 dz}
\end{aligned}$$

$$\begin{aligned}
&= (\omega_1\eta_2 - \omega_2\eta_1)\, dx\wedge dy \\
&\qquad + (\omega_2\eta_3 - \omega_3\eta_2)\, dy\wedge dz \\
&\qquad\qquad + (\omega_3\eta_1 - \omega_1\eta_3)\, dz\wedge dx
\end{aligned}$$

$$\therefore \quad \boxed{\begin{aligned}
\omega\wedge\eta = &(\omega_2\eta_3 - \omega_3\eta_2)\, dy\wedge dz \\
&+ (\omega_3\eta_1 - \omega_1\eta_3)\, dz\wedge dx \\
&\quad + (\omega_1\eta_2 - \omega_2\eta_1)\, dx\wedge dy
\end{aligned}}$$

- Linear combinations of $dy\wedge dz$, $dz\wedge dx$ & $dx\wedge dy$ are called <u>differential 2-forms</u> (on $\mathbb{R}^3$)

$$\boxed{\zeta = \zeta_1\, dy\wedge dz + \zeta_2\, dz\wedge dx + \zeta_3\, dx\wedge dy}$$

Similarly, if ω is a 1-form and ζ is a 2-form

then we can define $\omega\wedge\zeta$

eg: If $\omega = dx$, $\quad \zeta = dy \wedge dz$

then $\quad \omega \wedge \zeta = dx \wedge dy \wedge dz$

Note that we insist on the anti-commutativity of wedge product, we have

$$\begin{aligned} dx \wedge dy \wedge dz &= -dy \wedge dx \wedge dz \\ &= dy \wedge dz \wedge dx \\ &= -dz \wedge dy \wedge dx \\ &= dz \wedge dx \wedge dy \\ &= -dx \wedge dz \wedge dy \end{aligned}$$

And $\quad dx \wedge dx \wedge dy = \cdots = 0$ whenever one of the dx, dy, dz repeated.

Hence, as $\dim \mathbb{R}^3 = 3$, all "linear combinations" of "3-forms" are just $\quad f\, dx \wedge dy \wedge dz$

which is called a differential 3-form (also called a volume form if $f > 0$)

Note: It is convenient to call smooth functions f the differential 0-form.

Summary (on $\mathbb{R}^3$)

0-form	:	f
1-form	:	$\omega_1 dx + \omega_2 dy + \omega_3 dz$
2-form	:	$\xi_1 dy\wedge dz + \xi_2 dz\wedge dx + \xi_3 dx\wedge dy$
3-form	:	$g\, dx\wedge dy\wedge dz$.

where, f, g, ω_i, ξ_i are (smooth) functions

Note: One can certainly define k-form for any $k \geq 0$. But in $\mathbb{R}^3$, k-forms are zero for $k>3$:

$$dx^i \wedge dx\wedge dy\wedge dz = 0, \quad \text{where } dx^i = dx, dy, \text{or } dz.$$

Change of Variables Formula: ($\mathbb{R}^2$)

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\Rightarrow \begin{cases} dx = x_u\, du + x_v\, dv \\ dy = y_u\, du + y_v\, dv \end{cases}$$

$$\Rightarrow \quad dx\wedge dy = (x_u\, du + x_v\, dv)\wedge(y_u\, du + y_v\, dv)$$

$$= (x_u y_v - x_v y_u)\, du\wedge dv$$

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\wedge dv$$

$$\boxed{dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv}$$

← Jacobian determinant.

Hence naturally

$$\boxed{\iint f(x,y)\, dx \wedge dy = \iint f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv}$$

Compare with

$$\boxed{\iint f(x,y)\, dx\, dy = \iint f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv}$$

Similarly for $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$

$$\boxed{dx \wedge dy \wedge dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du \wedge dv \wedge dw} \quad (\text{Ex!})$$

(using $dx = x_u du + x_v dv + x_w dw, \cdots$)

- "Oriented" change of variables formula
- "$dx \wedge dy$" oriented area element
- "$dx \wedge dy \wedge dz$" oriented volume element.

(see later remark)

Exterior differentiation "d" on a form "ω".

0-form f	df (1-form)
1-form $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$	$d\omega = d\omega_1 \wedge dx + d\omega_2 \wedge dy + d\omega_3 \wedge dz$ (2-form)
2-form $\xi = \xi_1 dy \wedge dz + \xi_2 dz \wedge dx + \xi_3 dx \wedge dy$	$d\xi = d\xi_1 \wedge dy \wedge dz + d\xi_2 \wedge dz \wedge dx + d\xi_3 \wedge dx \wedge dy$ (3-form)
3-form $f\, dx \wedge dy \wedge dz$	$df \wedge dx \wedge dy \wedge dz = 0$ (4-form) in $\mathbb{R}^3$

eg.0 $d(dx) = d(dy) = d(dz) = 0$. ($d^2x = d^2y = d^2z = 0$)

eg.1 (in $\mathbb{R}^2$) $\omega = M\,dx + N\,dy$ ($M = M(x,y)$, $N = N(x,y)$)

then
$$\begin{aligned} d\omega &= dM \wedge dx + dN \wedge dy \\ &= (M_x dx + M_y dy) \wedge dx + (N_x dx + N_y dy) \wedge dy \\ &= (N_x - M_y)\, dx \wedge dy \end{aligned}$$

(+ve) oriented area

In this notation, Green's Thm $\oint_{C=\partial R} M dx + N dy = \iint_R (N_x - M_y)\, dx\, dy$,

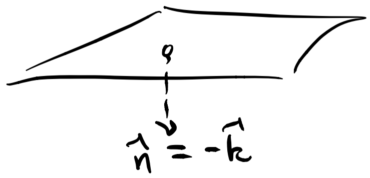
can be written as $\boxed{\oint_{C=\partial R} \omega = \iint_R d\omega}$

Remark: If we let $\vec{F} = M\hat{i} + N\hat{j} \leftrightarrow \omega = M\,dx + N\,dy$

then $(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dA = (N_x - M_y)\underbrace{\hat{k}\cdot\hat{n}\,dA}_{dx\wedge dy} = d\omega$

$(\hat{n} = \hat{k})$

Hence, if we use



then

$$\hat{k}\cdot\hat{n}\,dA = \begin{cases} dx\wedge dy & \text{if } \hat{n} = \hat{k} \\ dy\wedge dx & \text{if } \hat{n} = -\hat{k} \end{cases}$$

orientation of the "surface"

eg2: $\zeta = \zeta_1\,dy\wedge dz + \zeta_2\,dz\wedge dx + \zeta_3\,dx\wedge dy$

Then
$$\begin{aligned} d\zeta &= d\zeta_1\wedge dy\wedge dz + d\zeta_2\wedge dz\wedge dx + d\zeta_3\wedge dx\wedge dy \\ &= \left(\frac{\partial\zeta_1}{\partial x}dx + \cdots\right)\wedge dy\wedge dz \\ &\quad + \left(\cdots + \frac{\partial\zeta_2}{\partial y}dy + \cdots\right)\wedge dz\wedge dx \\ &\quad + \left(\cdots + \frac{\partial\zeta_3}{\partial z}dz\right)\wedge dx\wedge dy \end{aligned}$$

$$= \left(\frac{\partial\zeta_1}{\partial x} + \frac{\partial\zeta_2}{\partial y} + \frac{\partial\zeta_3}{\partial z}\right) dx\wedge dy\wedge dz$$

$$= \operatorname{div}\vec{F}\; dx\wedge dy\wedge dz$$

where $\vec{F} = \zeta_1\hat{i} + \zeta_2\hat{j} + \zeta_3\hat{k}$

Hence the divergence theorem can be written as:

$$\iiint_D d\zeta = \iiint_D \left(\frac{\partial \zeta_1}{\partial x}+\frac{\partial \zeta_2}{\partial y}+\frac{\partial \zeta_3}{\partial z}\right) \underbrace{dx\wedge dy\wedge dz}_{\text{(+ve) oriented volume}}$$

$$= \iiint_D \operatorname{div}\vec{F}\, dV = \iint_{S=\partial D} \vec{F}\cdot\hat{n}\, d\sigma$$

outward

To see the relation between $\vec{F}\cdot\hat{n}\, d\sigma$ and ζ,

we parametrize S:

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_u\times\vec{r}_v = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}\hat{i} + \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix}\hat{j} + \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\hat{k}$$

If $\vec{r}_u\times\vec{r}_v$ is <u>outward</u>, then

$$\hat{n} = \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|} \quad \text{and} \quad d\sigma = |\vec{r}_u\times\vec{r}_v|\, du\,dv$$

$$= |\vec{r}_u\times\vec{r}_v|\, du\wedge dv$$

(correct orientation)

then
$$\vec{F}\cdot\hat{n}\, d\sigma = \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{\cancel{|\vec{r}_u\times\vec{r}_v|}}\cancel{|\vec{r}_u\times\vec{r}_v|}\, du\wedge dv$$

$$= \left(\zeta_1\frac{\partial(y,z)}{\partial(u,v)} + \zeta_2\frac{\partial(z,x)}{\partial(u,v)} + \zeta_3\frac{\partial(x,y)}{\partial(u,v)}\right) du\wedge dv$$

$$= \zeta_1\, dy\wedge dz + \zeta_2\, dz\wedge dx + \zeta_3\, dx\wedge dy$$

$$= \zeta$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{(u,v)} \zeta_1\, dy\wedge dz + \zeta_2\, dz\wedge dx + \zeta_3\, dx\wedge dy$$

$$= \iint_{S=\partial D} \zeta$$

Hence divergence thm is

$$\boxed{\iiint_D d\zeta = \iint_{S=\partial D} \zeta}$$

ζ = 2-form

<u>eg3</u> <u>Stokes' Thm</u>

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \Leftrightarrow \omega = M\,dx + N\,dy + L\,dz$$

Then $d\omega = (L_y - N_z)\,dy\wedge dz + (M_z - L_x)\,dz\wedge dx$ (Ex!)
$\qquad + (N_x - M_y)\,dx\wedge dy$

$$= (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma \qquad (\text{Ex!})$$

<u>Stokes Thm becomes</u>

$$\oint_{C=\partial S} \vec{F}\cdot d\vec{r} \longrightarrow \boxed{\oint_{C=\partial S} \omega = \iint_S d\omega} \longleftarrow \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$$

Generalization to manifold of n-dimension with boundary (Skipped)

- M = n dim'l Manifold (oriented)
- ∂M = boundary (oriented with induced orientation)
- ω = $(n-1)$-form on M (smooth)

Then

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

$\int_M$: n-dim'l integral

$\int_{\partial M}$: $(n-1)$-dim'l integral

Note: ∂M is always closed, i.e. no boundary.

$$\therefore \boxed{\partial(\partial M) = \partial^2 M = 0}$$

boundary has no boundary

∂S is a closed curve

Hence if $\omega = d\eta$, for some $(n-2)$-form η, then

then $\int_M d(d\eta) = \int_M d\omega = \int_{\partial M} \omega$

$= \int_{\partial M} d\eta = \int_{\partial(\partial M)} \eta = 0$ (for any η.)

This suggests $\boxed{d^2\eta = 0}$, $\forall$ differential form

Ex: Verify this for 0-form and 1-form in $\mathbb{R}^3$ and observes that these are just

$$\begin{cases} \vec{\nabla} \times \vec{\nabla} f = 0 & (d^2 f = 0) \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0 & (d^2\omega = 0) \end{cases}$$

eg: Let $\omega = \dfrac{-y}{x^2+y^2}dx + \dfrac{x}{x^2+y^2}dy$

Check: $d\omega = 0$

But $\omega \neq df$ for any smooth function on $\mathbb{R}^2 \setminus \{(0,0)\}$

(Since $\omega = d\theta$ and θ is not defined on $\mathbb{R}^2 \setminus \{(0,0)\}$)

Hence $\quad d\omega = 0 \not\Rightarrow \omega = d\eta$ in general

$(\Leftarrow)$

yes

Note: Thm 10 can be written as:

$\Omega \subset \mathbb{R}^2$ simply-connected, then

$d\omega = 0 \Leftrightarrow \omega = df$ for some smooth function f on Ω.