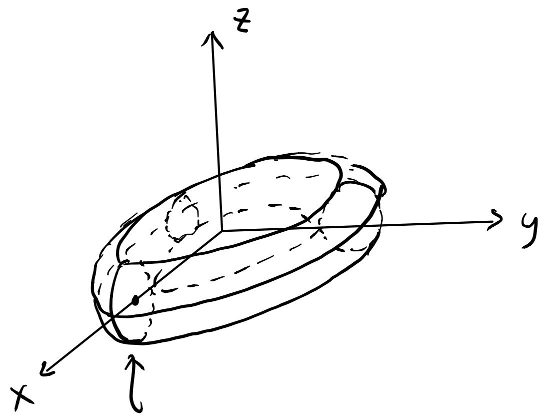
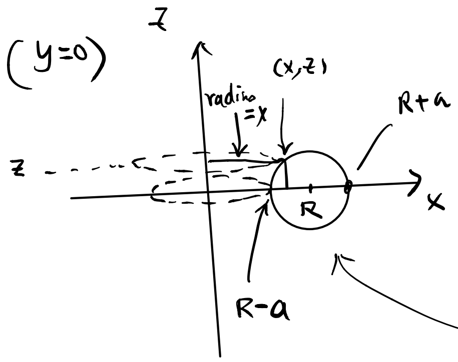
eg51



a circle in xz-plane

rotating this circle around the z-axis gives "torus"



For $y=0$ (i.e. xz-plane)

the circle can be parametrized by

$$\begin{cases} x = R + a\cos\alpha \\ z = a\sin\alpha \end{cases} \qquad 0 \le \alpha \le 2\pi \quad (R > a > 0)$$

Revolving around the z-axis, we have

$$\begin{cases} x = (R + a\cos\alpha)\cos\theta \\ y = (R + a\cos\alpha)\sin\theta \\ z = a\sin\alpha \end{cases} \qquad \begin{array}{l} 0 \le \alpha \le 2\pi \\ 0 \le \theta \le 2\pi \\ (R > a > 0) \end{array}$$

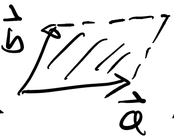
is a parametrization of the torus.

Note that this torus can also be described as

$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = a^2 \qquad (\text{Ex!})$$

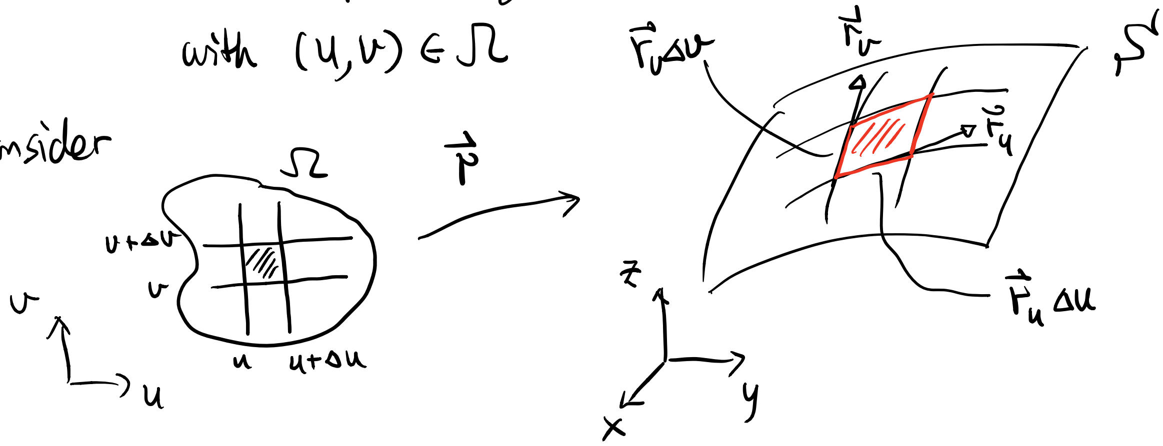
Surface Area

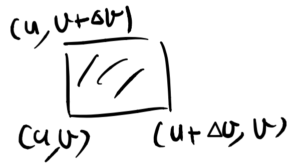
Recall: for $\vec{a}, \vec{b} \in \mathbb{R}^3$

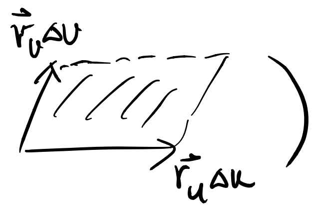
$|\vec{a} \times \vec{b}| = \text{Area}\Big($  $\Big)$

Let $\vec{r}(u,v)$ be a parametrization of a surface S with $(u,v) \in \Omega$

Consider



$\Rightarrow$ "Area" on the surface corresponding to 

is approx. $= \text{Area}\Big($  $\Big)$

$$= |(\vec{r}_u \Delta u) \times (\vec{r}_v \Delta v)|$$

$$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

Hence "Area element" of S, denoted by $d\sigma$, is given by

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

$$\boxed{d\sigma = |\vec{r}_u \times \vec{r}_v| \, dA}$$

dA: area element in the (u,v)-space.

Therefore, we make the following

Def 15: Let $S \subset \mathbb{R}^3$ be a smooth parametric surface given by $\vec{r}(u,v)$ for $(u,v) \in \Omega \subset \mathbb{R}^2$. Then

$$\text{Area}(S) \stackrel{\text{def}}{=} \iint_{\Omega} |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_{\Omega} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$

(i.e. $\text{Area}(S) = \iint_{\Omega} d\sigma$)

eg 52: Surface area of torus given by ($R > a > 0$ are constants)

$$\begin{cases} x = (R + a\cos\alpha)\cos\theta \\ y = (R + a\cos\alpha)\sin\theta \\ z = a\sin\alpha \end{cases} \qquad \begin{matrix} 0 \le \alpha \le 2\pi \\ 0 \le \theta \le 2\pi \end{matrix}$$

i.e. $\vec{r}(\alpha,\theta) = (R + a\cos\alpha)\cos\theta \, \hat{i} + (R + a\cos\alpha)\sin\theta \, \hat{j} + a\sin\alpha \, \hat{k}$

$$\Rightarrow \begin{cases} \dfrac{\partial \vec{r}}{\partial \alpha} = -a\sin\alpha\cos\theta \, \hat{i} - a\sin\alpha\sin\theta \, \hat{j} + a\cos\alpha \, \hat{k} \\[2ex] \dfrac{\partial \vec{r}}{\partial \theta} = -(R + a\cos\alpha)\sin\theta \, \hat{i} + (R + a\cos\alpha)\cos\theta \, \hat{j} \end{cases}$$

&

$$\frac{\partial \vec{r}}{\partial \alpha} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\alpha\cos\theta & -a\sin\alpha\sin\theta & a\cos\alpha \\ -(R + a\cos\alpha)\sin\theta & (R + a\cos\alpha)\cos\theta & 0 \end{vmatrix}$$

(check) $-a(R+a\cos\alpha)\cos\theta\cos\alpha\,\hat{i} - a(R+a\cos\alpha)\sin\theta\cos\alpha\,\hat{j}$

$- a(R+a\cos\alpha)\sin\alpha\,\hat{k}$

$$\left|\frac{\partial \vec{r}}{\partial \alpha}\times\frac{\partial \vec{r}}{\partial \theta}\right| = a(R+a\cos\alpha)\left[\cos^2\theta\cos^2\alpha + \sin^2\theta\cos^2\alpha + \sin^2\alpha\right]^{1/2}$$

$$= a(R+a\cos\alpha)$$

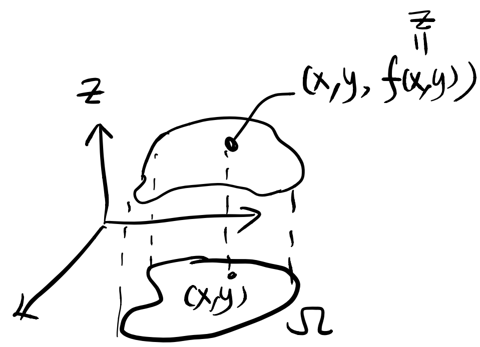
Hence $\text{Area}(\text{Torus}) = \iint_{\Omega} \left|\frac{\partial \vec{r}}{\partial \alpha}\times\frac{\partial \vec{r}}{\partial \theta}\right| dA$

$$= \int_0^{2\pi}\int_0^{2\pi} a(R+a\cos\alpha)\, d\alpha\, d\theta$$

(check) $= 4\pi^2 Ra$ #

Surface area of a graph

$z = f(x,y)$, $(x,y)\in\Omega$



Choose the following "nature" parametrization of the graph

$$\vec{r}(x,y) = x\,\hat{i} + y\,\hat{j} + f(x,y)\,\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_x = \hat{i} + f_x\,\hat{k} \\ \vec{r}_y = \hat{j} + f_y\,\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\,\hat{i} - f_y\,\hat{j} + \hat{k}$$

$$\Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{1 + |\nabla f|^2}$$

$$\geq 1 \quad (\text{non-zero, hence "smooth" if } f \in C^1)$$

Thm 11: The surface area of a C^1 graph S given by

$$z = f(x,y), \quad (x,y) \in \Omega \subset \mathbb{R}^2$$

is

$$\text{Area}(S) = \iint_\Omega \sqrt{1 + |\nabla f|^2}\, dA = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dA$$

(Similarly for $x = f(y,z)$ or $y = f(x,z)$)

Implicit Surface (Level surface)

Suppose S is given by $F(x,y,z) = c$

i.e. $S = F^{-1}(c)$

(Note: F is a function of 3-variables, not vector field)

eg 53: $F(x,y,z) = x^2 + y^2 + z^2$

Is $F^{-1}(0)$ a surface?

No, since $F^{-1}(0) = \{(0,0,0)\}$, not a surface!

Remark: If $\vec{\nabla} F \neq 0$ at a point, then IFT implies that $S = F^{-1}(c)$ is a "surface" (c = value of F at that point) near that point (in fact, a graph!)

eg 53 (cont'd) $\vec{\nabla} F = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\therefore \quad \vec{\nabla} F = 0 \Leftrightarrow (x,y,z) = (0,0,0)$

Hence if $c > 0$, then $\forall (x,y,z) \in F^{-1}(c)$, we have

$$\vec{\nabla} F(x,y,z) \neq 0 \quad \left(\text{since } x^2+y^2+z^2 = c > 0 \atop \Rightarrow (x,y,z) \neq (0,0,0)\right)$$

$\Rightarrow \quad S = F^{-1}(c) \ (\forall c > 0)$ is a surface.

(What are these surfaces?)

Terminology: $S = F^{-1}(c)$ is said to be smooth

if (1) F is C^1 on S, and

(2) $\vec{\nabla} F \neq 0$ on S.

How to compute surface area for a smooth level surface

$S = F^{-1}(c)$?