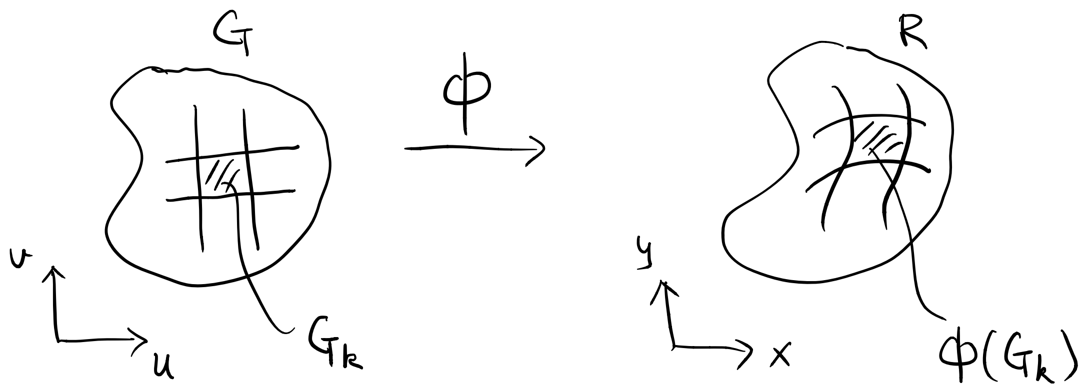
General change of coordinate formula in $\mathbb{R}^2$

Suppose $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ denoted by

$$\phi(u,v) = (x,y), \quad \phi : G^{(\subset uv\text{-plane})} \to R^{(\subset xy\text{-plane})}$$



Idea: We need to find

$$\frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \to ? \quad \text{as "} G_k \to \text{point"}$$

If ϕ is a diffeomorphism : 1-1, onto & $\phi, \phi^{-1} \in C^1$.

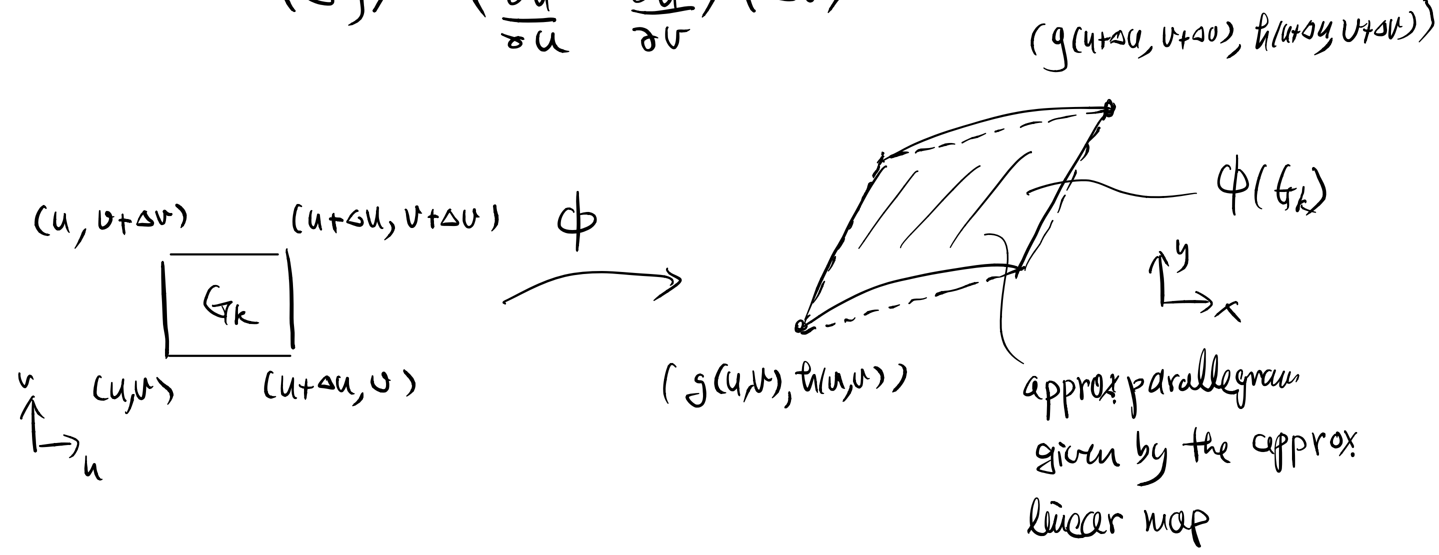
ϕ is $C^1 \Rightarrow$

$$\begin{cases} g(u+\Delta u, v+\Delta v) = g(u,v) + \frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v + \cdots \\ h(u+\Delta u, v+\Delta v) = h(u,v) + \frac{\partial h}{\partial u}\Delta u + \frac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

$$\Rightarrow \begin{cases} \Delta x = \Delta g = g(u+\Delta u, v+\Delta v) - g(u,v) = \frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v + \cdots \\ \Delta y = \Delta h = h(u+\Delta u, v+\Delta v) - h(u,v) = \frac{\partial h}{\partial u}\Delta u + \frac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

In matrix form

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + \cdots$$



(By linear algebra)

$$\frac{dA_{(x,y)}}{dA_{(u,v)}} \cong \frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \cong \left| \det \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} \right|$$

$$\cong \quad \left(\frac{\text{“}\Delta x \Delta y\text{”}}{\text{“}\Delta u \Delta v\text{”}} \right) \qquad = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

<u>Def 7</u>: Define the <u>Jacobian</u> $J(u,v)$ of the "coordinates" transformation

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

by

$$J(u,v) \overset{\text{notation}}{=\!=} \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

With this notation, we should have the formula

$$\boxed{\begin{aligned}\iint_{(x,y)\in R} f(x,y)\,dx\,dy &= \iint_{(u,v)\in G} f(g(u,v), h(u,v)) \left|\det\begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v}\end{pmatrix}\right| du\,dv \\ &= \iint_G f(x(u,v), y(u,v)) \left| J(u,v) \right| du\,dv \\ &= \iint_G f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv\end{aligned}}$$

eg 28 : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad \big((u,v) = (r,\theta) \big)$

$$\Rightarrow \quad J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{pmatrix} = r \quad \text{(check!)}$$

and $$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \iint_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

(same formula as before.)

Thm 6: Suppose $\phi: \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ is a diffeomaphism (1-1, onto, s.t. ϕ and $\phi^{-1} \in C^1$) mapping a region G (closed and bounded) in the uv-plane onto a region R (closed and bounded) in the xy-plane (except possibly on the boundary). Suppose $f(x,y)$ is continuous on R, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f \circ \phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Notes: (i) $f \circ \phi(u,v) = f(x(u,v), y(u,v))$

(ii) ϕ is a diffeomaphism $\Rightarrow$ $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| \neq 0$.

Triple integrals ("substitutions" in triple integrals)

$$\phi(u,v,w) = (x,y,z) : G^{\subset \mathbb{R}^3 \ni (u,v,w)} \longrightarrow D^{\subset \mathbb{R}^3 \ni (x,y,z)}$$

with $\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$ 1-1, onto, cont. differentiable and inverse also cont. differentiable.

Def 8 Jacobian (determinant) of transformation in $\mathbb{R}^3$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

$$\left[= \det \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \\ \frac{\partial k}{\partial u} & \frac{\partial k}{\partial v} & \frac{\partial k}{\partial w} \end{pmatrix} \right]$$

Note: Chain rule $\Rightarrow$

$$\begin{cases} \text{2-dim} & \dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(s,t)} = \dfrac{\partial(x,y)}{\partial(s,t)} \\ \text{3-dim.} & \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \dfrac{\partial(u,v,w)}{\partial(s,t,r)} = \dfrac{\partial(x,y,z)}{\partial(s,t,r)} \end{cases}$$

(Ex!)

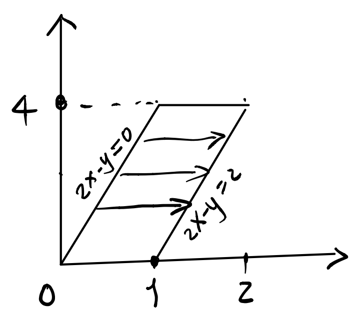
$\Rightarrow$

$$\begin{cases} \text{2-dim} & \dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}} \\ \text{3-dim} & \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{\dfrac{\partial(x,y,z)}{\partial(u,v,w)}} \end{cases}$$

(Ex!)

$$\boxed{\begin{aligned} &\text{Thm 7: Under similar conditions of Thm 6} \\ &\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G F\circ\phi(u,v,w)\,\left|J(u,v,w)\right|\,du\,dv\,dw \\ &= \iiint_G F\big(g(u,v,w), h(u,v,w), k(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \end{aligned}}$$

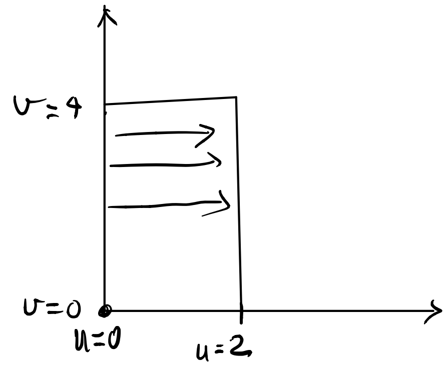
eg 29
$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$$



lower limit $x = \frac{y}{2} \Leftrightarrow 2x-y=0$

upper limit $x = \frac{y}{2}+1 \Leftrightarrow 2x-y=2$

Define $\begin{cases} u = 2x - y \\ v = y \end{cases}$



Then $\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = v \end{cases}$

$\begin{cases} 2x - y = 0 \leftrightarrow u = 0 \\ 2x - y = 2 \leftrightarrow u = 2 \end{cases}$

$\begin{cases} y = 0 \\ y = 4 \end{cases} \leftrightarrow \begin{matrix} v = 0 \\ v = 4 \end{matrix}$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

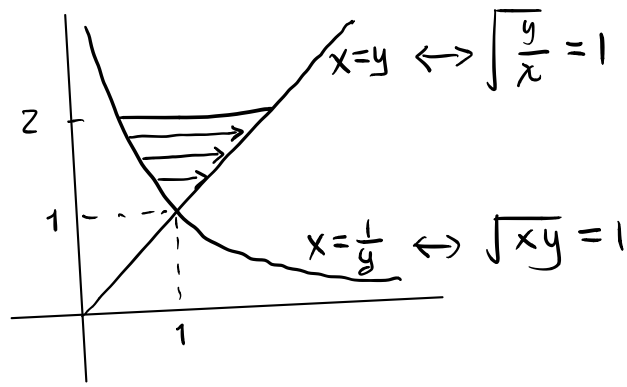
$$= \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \frac{1}{2}$$

$$\therefore \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy = \int_0^4 \int_0^2 \frac{u}{2} \left|\frac{1}{2}\right| du\,dv = 2 \quad (\text{check!})$$

$\#$

<u>eg30</u> $I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\,dy$

Domain of integration

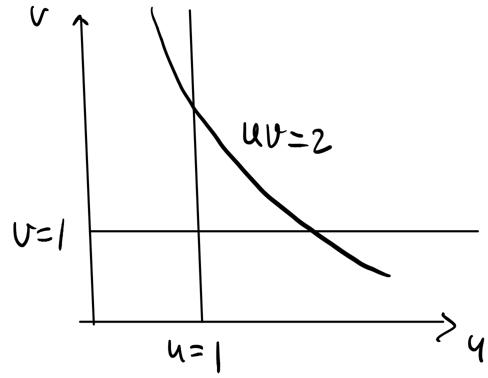


Let $\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases}$

(this should simplify the integration)

Then "boundary curves"

$$\begin{cases} x = y \longleftrightarrow v = 1 \\ x = \frac{1}{y} \longleftrightarrow u = 1 \\ y = 2 \longleftrightarrow uv = 2 \end{cases}$$

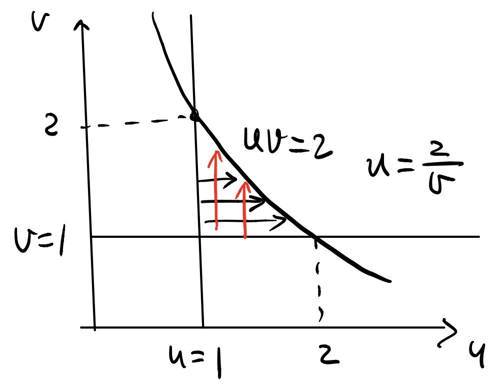


And the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Express x, y in terms of $u\,v$: $\begin{cases} x = \frac{u}{v} \\ y = uv \end{cases}$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix} = \frac{2u}{v} \quad (\text{check!})$$



(u>0
v>0)

$$I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

$$= \int_1^2 \int_1^{\frac{2}{v}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$\left(\text{or} = \int_1^2 \int_1^{\frac{2}{u}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\, du \right)$$

Let do $\displaystyle\int_1^2 \int_1^{\frac{2}{u}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\, du = \int_1^2 \int_1^{\frac{2}{u}} v e^u \cdot \frac{2u}{v}\, dv\, du$

$$= \int_1^2 2ue^u \left(\int_1^{\frac{2}{u}} dv\right) du \qquad (\text{Check!})$$

$$= \int_1^2 2ue^u \left(\frac{2}{u} - 1\right) du = 2e(e-2) \quad \#$$

eg18 revisit Volume of Ellipsiod

$$D = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \quad (a, b, c > 0)$$

$$\mathrm{Vol}(D) = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

Change of variables

$$\begin{cases} u = \frac{x}{a} \\ v = \frac{y}{b} \\ w = \frac{z}{c} \end{cases} \Leftrightarrow \begin{cases} x = au \\ y = bv \\ z = cw \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix} = abc \ (>0)$$

$$\mathrm{Vol}(D) = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

$$= 8 \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dw\, dv\, du$$

(since D transforms to the solid unit ball $u^2+v^2+w^2 \le 1$)

$$= abc \cdot 8 \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} dw\, dv\, du$$

$$= abc \cdot \mathrm{Vol}(\text{Solid unit ball in } (u,v,w)\text{-space})$$

$$= abc \cdot \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$ ((ρ, ϕ, θ) = spherical coordinates in the (u,v,w)-space)

$$= \frac{4\pi}{3} abc$$

$\divideontimes$

eg31 Let $D = \{(x,y,z) \in \mathbb{R}^3 : |x|+|y|+|z| \le 1\}$

Evaluate $\iiint_D (x+y+z)^4 \, dV$.

(can use symmetric $(x,y,z) \leftrightarrow (-x,-y,-z)$ to reduce half, but not to the 1st octant

since for instance $x+y+z \leftrightarrow x+y-z$

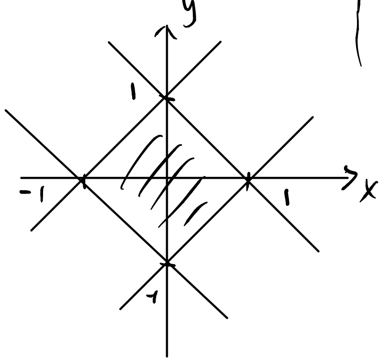
under $(x,y,z) \leftrightarrow (x,y,-z)$

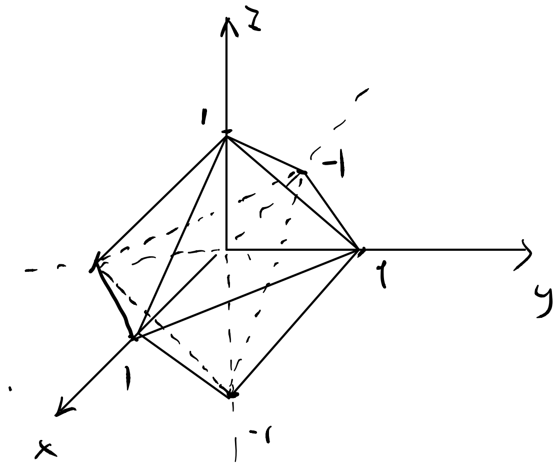
$(x+y+z)^4$ is not symmetric in all reflection with respect to the coordinates lines)

Soln: If $z=0$,

then $|x|+|y| \le 1$

$$\left\{ \begin{array}{l} x+y = \pm 1 \\ x-y = \pm 1 \end{array} \right.$$





Boundary surfaces are given by

$\pm x \pm y \pm z = 1$ (8 surfaces)

Let
$$\begin{cases} u = x+y+z \\ v = x+y-z \\ w = x-y-z \end{cases}$$

boundary planes
$$\begin{cases} u = \pm 1 \\ v = \pm 1 \\ w = \pm 1 \end{cases}$$
only 6 out of 8 surfaces.

need to find formula for other 2 boundary surfaces.