7. Mobius strip or band

$[0,1]\times[0,1]/\sim$ where $(s_1,s_2)\sim(t_1,t_2)$ if

$(s_1,s_2)=(t_1,t_2)$ or $\begin{cases} |s_1-t_1|=0,1 \\ s_2=1-t_2 \end{cases}$

For simplicity, often say

identify $(0,t)$ with $(1,1-t)$ on $[0,1]^2$

8. Klein Bottle

Identify $(s,0)$ with $(s,1)$ and

$(0,t)$ with $(1,1-t)$ on $[0,1]^2$

Note. Klein Bottle $\not\subset \mathbb{R}^3$

But basic neighborhoods of any point

$\|$ homeo.

$\{z\in\mathbb{C} : |z|<1\}$

9. Projective Plane, $\mathbb{RP}^2$

Identify $(s,0)$ with $(1-s,1)$ and

$(0,t)$ with $(1,1-t)$ on $[0,1]^2$

OR

Identify z with $-z$ if $|z|=1$ on

$\{z\in\mathbb{C} : |z|\le 1\}$

Exercise. Show they are homeomorphic.

Given $(X, \mathcal{T}_X)$ and $\sim$ on X

$q : X \xrightarrow{\text{onto}} X/\sim$ or any Q

$\mathcal{T}_q = \{ V \subset X/\sim \text{ or } Q : q^{-1}(V) \in \mathcal{T}_X \}$

QT1 $q : (X, \mathcal{T}_X) \longrightarrow (X/\sim \text{ or } Q, \mathcal{T}_q)$ is continuous

Obvious, because if $V \in \mathcal{T}_q$,

by definition, $q^{-1}(V) \in \mathcal{T}_X$

QT2 $\mathcal{T}_q$ is the maximal topology on $X/\sim$ or Q

to make $q : (X, \mathcal{T}_X) \longrightarrow X/\sim$ or Q cts.

Suppose $q : (X, \mathcal{T}_X) \longrightarrow (X/\sim, \mathcal{T}')$ is continuous

Let $V \in \mathcal{T}'$, Then $q^{-1}(V) \in \mathcal{T}_X$

By def, $V \in \mathcal{T}_q$. $\therefore \mathcal{T}' \subset \mathcal{T}_q$

QT3 $f : X/\sim \text{ or } Q \longrightarrow Z$ is continuous

$\Leftrightarrow$ $f \circ q : X \longrightarrow Z$ is continuous

"$\Rightarrow$" Trivial

"$\Leftarrow$" Easy. Basically $(f \circ q)^{-1} = q^{-1} \circ f^{-1}$

QT4 $\mathcal{T}_q$ is the minimal topology on $X/\sim$ or Q

to make QT3 true.

Exercise.

Disjoint Union

Put two copies of X together $\neq X \cup X$

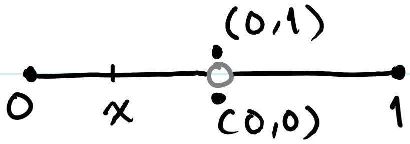
Define $X \sqcup X = (X \times \{0\}) \cup (X \times \{1\})$

A useful example

Let $X = [-1,1] \sqcup [-1,1]$

Identify $(x,0)$ with $(x,1)$ if $x \neq 0$

Picture



Qu. Is $X/\sim$ Hausdorff ?

Any nbhds of $(0,0)$ & $(0,1)$ intersect !

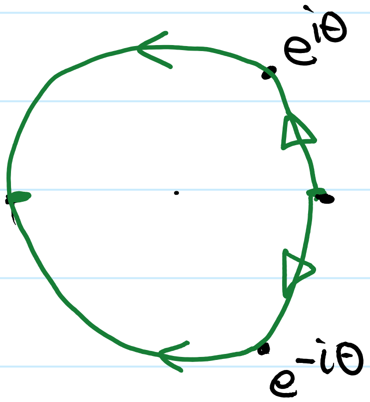
Sphere

Given $\mathbb{D}^2 = \{ z \in \mathbb{C} : |z| \leq 1 \} \subset \mathbb{R}^2$, standard

1. Identify $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ to one point

   i.e. $z \sim w$ if $|z| = |w| = 1$

2. Identify $e^{i\theta}$ with $e^{-i\theta}$ for all θ



$\rightsquigarrow$ "purse"

Attaching Space

Given X, Y ; $A \subset X$; $f: A \longrightarrow Y$

Define $X \cup_f Y$ by $(X \sqcup Y)/\sim$ where

$(a, 0)$ is identified with $(f(a), 1)$, $a \in A$

Usually, we say:

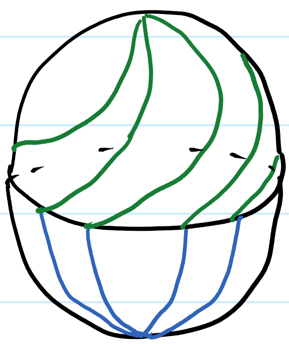
Attach X to Y along A by f

(i) $X = \mathbb{D}^2 = Y$, $A = \mathbb{S}^1$, $f = id: \mathbb{S}^1 \longrightarrow \mathbb{S}^1$

$X \cup_f Y = \mathbb{S}^2$, just like N-S hemisphere

(ii) $X = \mathbb{D}^2 = Y$, $A = \mathbb{S}^1$, $f(e^{i\theta}) = e^{i(\theta+\alpha)}$

$X \cup_f Y = \mathbb{S}^2$



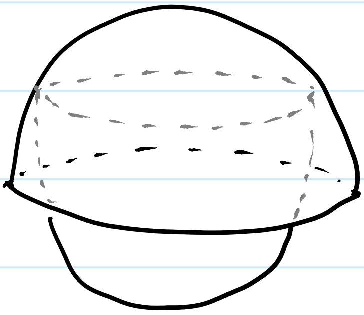
(iii) $X = \mathbb{D}^2 = Y$, $A = \mathbb{S}^1$

$f(e^{i\theta}) = e^{2i\theta}$

$X \cup_f Y$ is not a surface

(iv) $X = \mathbb{D}^2 = Y$, $A = \mathbb{S}^1$

$f(e^{i\theta}) = \frac{1}{2} e^{i\theta}$



Handle Body

(i) $X = S^2 \setminus (D_1 \cup D_2)$ where $D_1 = D_2 = D^2$

$Y = S^1 \times [0,1]$

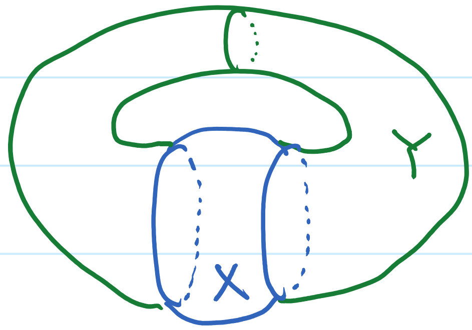
$A = \partial X = S_1 \cup S_2$ where $S_1 = S_2 = S^1$

Note that $\partial Y = S^1 \times \{0\} \cup S^1 \times \{1\}$

Let $f : A \longrightarrow \partial Y$

be a homeomorphism

Then $X \cup_f Y$ = Torus



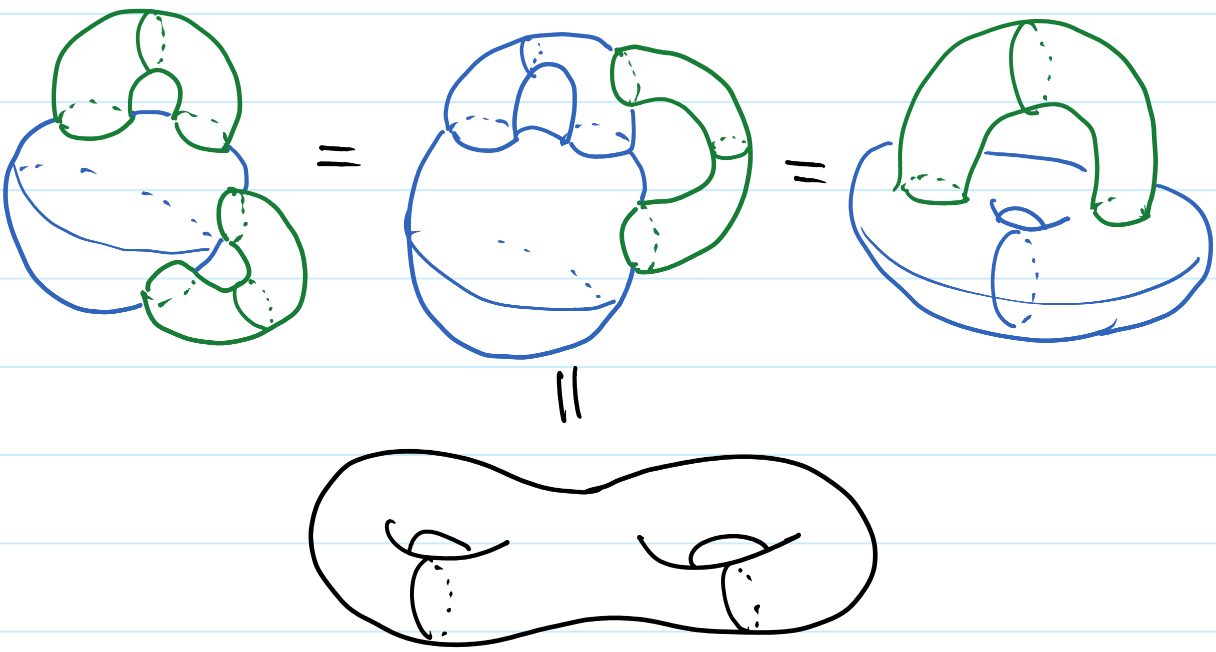
Usual say,

Attach a handle to a sphere

(ii) Attach two handles to a sphere

$\parallel$

Attach a handle to a torus



Projective Plane $\mathbb{RP}^2$

1. Identify $(s,0) \sim (1-s,1)$ and $(0,t) \sim (1,1-t)$
   on $[0,1] \times [0,1]$
2. Idenify $e^{i\theta} \sim -e^{i\theta}$ on $D^2 = \{ |z| \leq 1 \}$

3. Let $X = \mathbb{R}^3 \setminus \{0\}$ and $x \sim y$ if
   $\exists\ \lambda \neq 0$ such that $x = \lambda y$
   i.e., $x, y, 0$ are on a straight line
   $X/\sim$ becomes the space of lines in $\mathbb{R}^3$

4. Let $S^2 = \{ u \in \mathbb{R}^3 : \|u\| = 1 \}$
   Then $u, -u$ are called antipodal points
   In fact, $u, 0, -u$ form a diameter

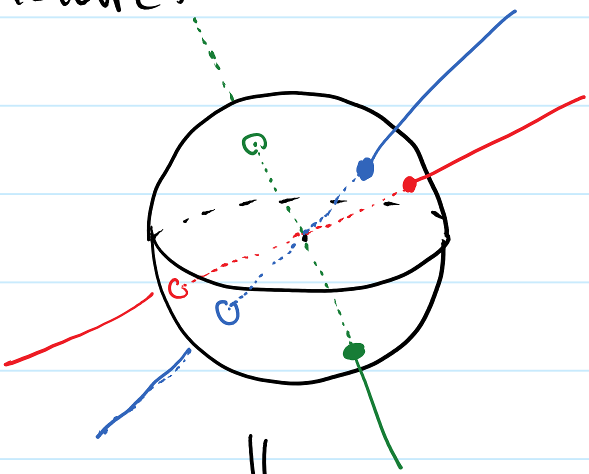
Fact. $S^2 /_{\text{antipodal points}} = \mathbb{R}^3 \setminus \{0\} /_{\text{st. lines}}$

$$[u] \longmapsto [u]$$

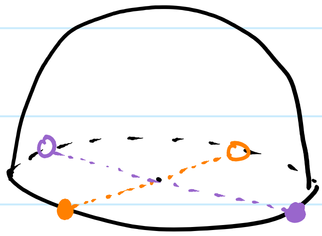
$$\left[\frac{x}{\|x\|} \right] \longleftarrow\!\shortmid [x]$$

Exercise. Show that it is a homeomorphism

5. Pictures



$=$



$$\frac{\mathbb{R}^3\setminus\{0\}}{\text{lines}} = S^2/_{\text{antipodal}} = \mathbb{D}^2/_{e^{i\theta}\sim -e^{i\theta}}$$

$$[x] \longmapsto [\pi(x)]$$

where $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$(x_1,x_2,x_3) \longmapsto (x_1,x_2,0)$$

Exercise. Show that it is a homeomorphism

6. Let $M_n(\mathbb{R}) = \{ n\times n \text{ matrices over } \mathbb{R} \}$

$$M_n(\mathbb{R}) \longrightarrow M_{n+1}(\mathbb{R}) : A \longmapsto \begin{bmatrix} 1 & 0\cdots 0 \\ 0 & \\ \vdots & A \\ 0 & \end{bmatrix}$$

$$O_n(\mathbb{R}) = \{ n\times n \text{ orthogonal matrices} \}$$

$$= \{ Q \in M_n(\mathbb{R}) : Q^TQ = QQ^T = I \}$$

Denote $O_3/O_2 = \{ \text{left cosets} \}$

$$= \{ A\cdot O_2 : A \in O_3 \}$$

In O_3/O_2, $A \cdot O_2 = B \cdot O_2$ if $A^{-1}B \in O_2$

In other words, $A \sim B$ on O_3 if $A^{-1}B \in O_2$

i.e. $$A^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{Q}} \\ 0 & & \end{bmatrix}$$ where $Q \in O_2$

Thus, $A^{-1}B(e_1) = e_1$ or $A(e_1) = B(e_1)$

Also, $A^{-1}B\big|_{0 \times \mathbb{R}^2} : 0 \times \mathbb{R}^2 \longrightarrow 0 \times \mathbb{R}^2$

is an isometry (given by Q)

Anyway, $O_3/O_2 \longrightarrow S^2$ by

$A \cdot O_2 \longmapsto A(e_1)$ is well-defined

Exercise. This is a homeomorphism

Now, denote $$\pm O_2 = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{Q}} \\ 0 & & \end{bmatrix} : Q \in O_2 \right\}$$

Then, only difference, if $A \sim B$

then $A(e_1) = \pm B(e_1)$

And $O_3/\pm O_2 \longrightarrow \mathbb{R}P^2$

$A \cdot (\pm O_2) \longmapsto [\pm A(e_1)]$

Projective Space $\mathbb{R}P^n = O_{n+1}/\pm O_n$

"n-dim spaces in $\mathbb{R}^{n+1}$"