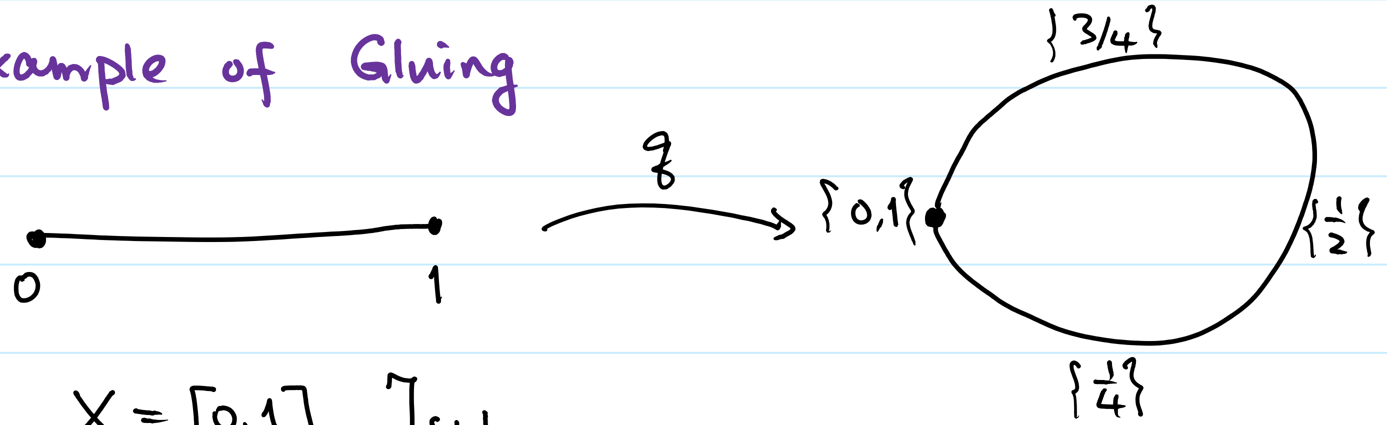
Example of Gluing



$X = [0,1]$, $\mathcal{T}_{std}$

As a set, we can see the "circle" as

$X/\sim$ where $\sim$ is an equiv. relation.

For $s, t \in [0,1]$, $s \sim t$ if $|s-t| = 0, 1$

$s = t$ or $\begin{matrix} s=0,\ t=1 \\ s=1,\ t=0 \end{matrix}$

In such a case,

$$X/\sim \;= \{ \underset{[0]=[1]}{\{0,1\}},\ \underset{[x]}{\{x\}}, 0 < x < 1 \}$$

Qu. How to put a topology on $X/\sim$?

The relation $\sim$ is equivalent to

$$q : X \longrightarrow X/\sim \;:\; x \longmapsto [x]$$

Natural expectation

$$(X, \mathcal{T}_X) \xrightarrow{\ q\ } (X/\sim, ?) \quad \text{continuous}$$

$$\mathcal{T}_q = \{ V \subset X/\sim \;:\; q^{-1}(V) \in \mathcal{T}_X \}$$

Exercise: Verify that it is a topology.

Quotient Topology

Given $(X, \mathcal{T}_X)$, either $\sim$ or $q : X \xrightarrow{\text{onto}} Q$, $Q = X/\sim$

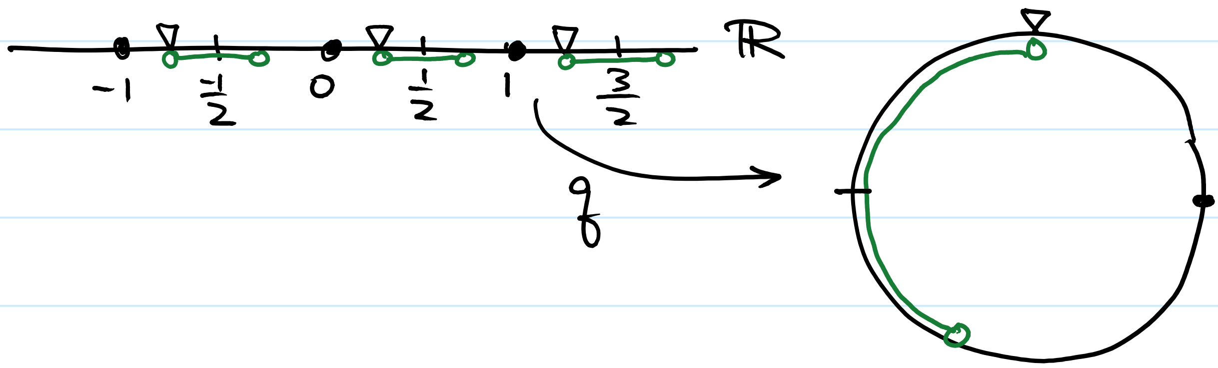
The quotient topology $\mathcal{T}_q$ on $X/\sim$ or Q

$$\mathcal{T}_q = \{ V \subset X/\sim \; : \; q^{-1}(V) \in \mathcal{T}_X \}$$

Circles

1. circle as $[0,1]/\sim$
2. $X = \mathbb{R}$, $\mathcal{T}_X = \mathcal{T}_{std}$

   $x \sim y$ if $x - y \in \mathbb{Z}$



3. All the above are the "circle"

$$[0,1]/\sim \;\xleftrightarrow[\text{homeo.}]{}\; \mathbb{R}/\mathbb{Z} \;\xleftrightarrow[\text{homeo}]{}\; S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$$

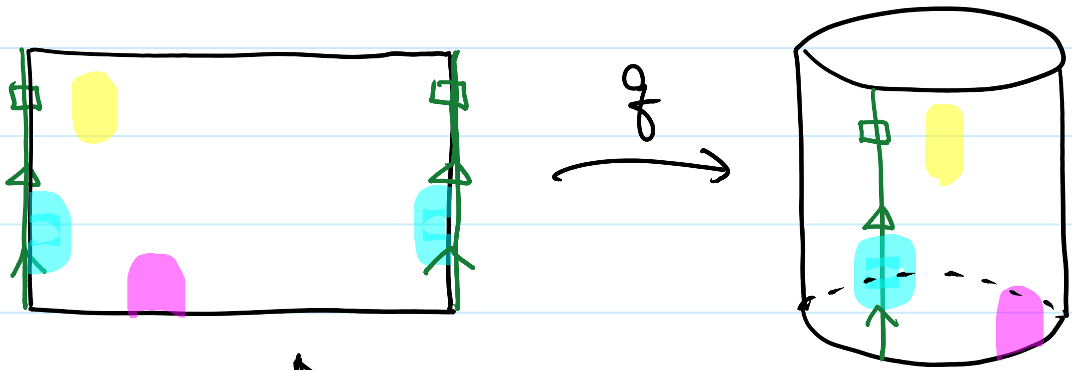
$$[x] \longmapsto e^{2\pi i x}$$

4. Similarly, we have cylinder

$\left([0,1]\times[0,1]\right)/\sim$ where

$$(s_1,s_2)\sim(t_1,t_2) \text{ if } \begin{cases} |s_1-t_1|=0,1 \\ s_2=t_2 \end{cases}$$

Gluing only on the 1st coordinate



$[0,1]\times[0,1] \overset{\text{homeo}}{\longleftrightarrow} \dfrac{\mathbb{R}\times[0,1]}{\mathbb{Z}\times 0} \overset{\text{homeo}}{\longleftrightarrow} S^1\times[0,1]\subset\mathbb{R}^3$ $=$ cylinder

5. Torus

Recall that it can be seen as

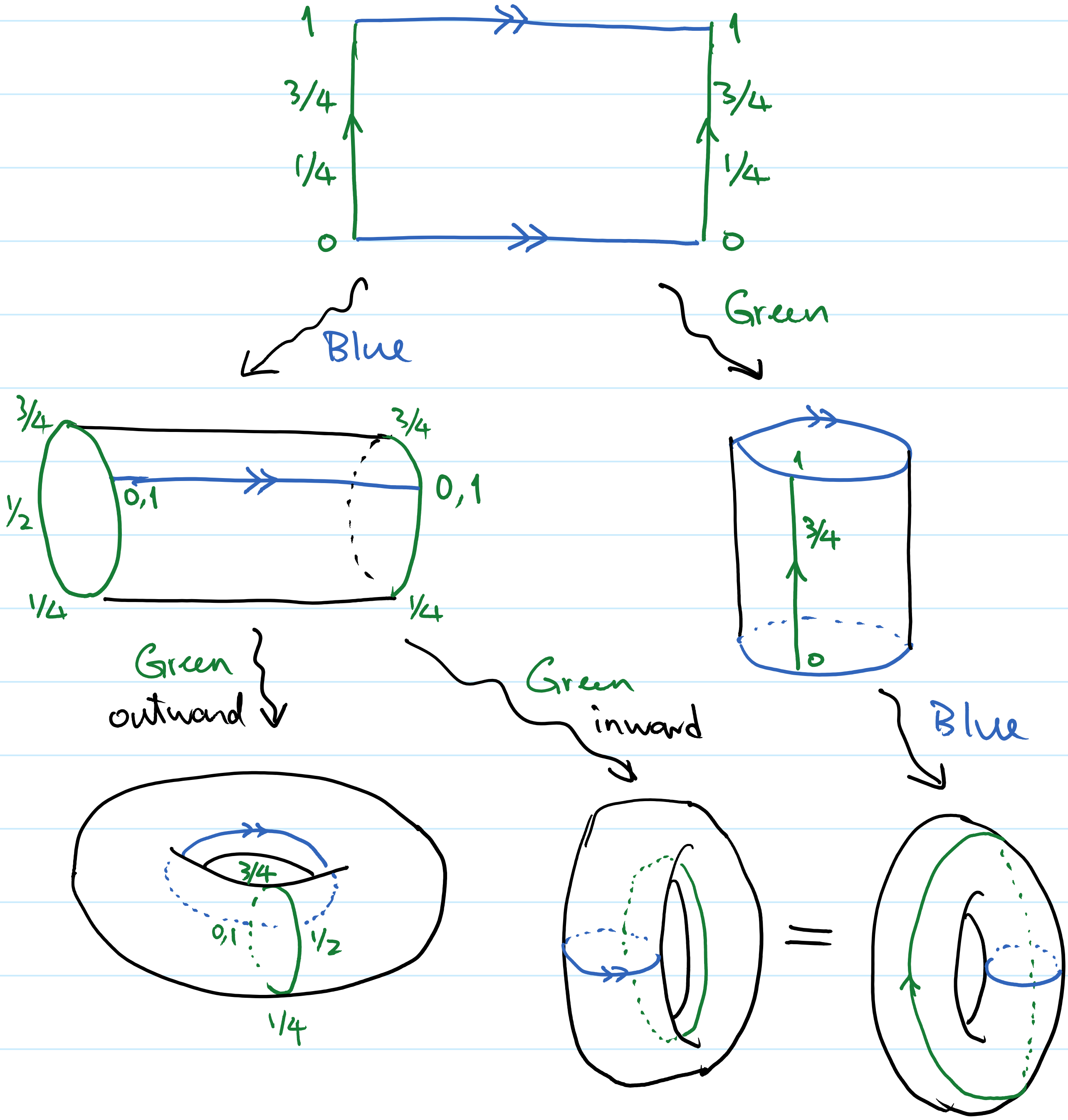
* surface of revolution $\subset \mathbb{R}^3$
* $S^1\times S^1$, product topology

$\left([0,1]\times[0,1]\right)/\sim$ where

$$(s_1,s_2)\sim(t_1,t_2) \text{ if } \begin{cases} |s_1-t_1|=0,1\,, \\ |s_2-t_2|=0,1 \end{cases}$$

$\updownarrow$

$\mathbb{R}^2/\mathbb{Z}^2$



Note that from above pictures, apparently the result depends on how it is glued. In fact, all are homeomorphic, just "placed" differenty in $\mathbb{R}^3$.

6. $\mathbb{R}^n/\mathbb{Z}^n = S^1 \times \cdots \times S^1$, n-dim Torus