Lect08-0216-Convergence

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Recall two logical statements

Continuity of f at x_0

$\boxed{\forall V \in \mathcal{T}_Y}$ with $f(x_0) \in V$, $\exists\, U \in \mathcal{T}_X$, $x_0 \in U$

$U \subset f^{-1}(V)$, i.e., $f(U) \subset V$

A cluster point x_0 of A

$\boxed{\forall U \in \mathcal{T}_X}$ with $x_0 \in U$, $U \cap A \setminus \{x_0\} \neq \emptyset$

The $\mathcal{T}_Y$ or $\mathcal{T}_X$ can be replaced by bases

In fact, local bases at $f(x_0)$ or x_0.

In the cases of metric space, can be

$\forall\ 1 \le n \in \mathbb{Z}$ $B_Y(f(x_0), \frac{1}{n})$ or $B_X(x_0, \frac{1}{n})$

Example. Cluster point in metric space

$\forall\ 1 \le n \in \mathbb{Z}$ $\underbrace{B(x_0, \tfrac{1}{n}) \cap A \setminus \{x_0\} \neq \emptyset}$

$\exists\ a_n \in B(x_0, \frac{1}{n})$, $a_n \in A$, $a_n \neq x_0$

We actually have a situation:

local base $\mathcal{U}_{x_0} = \{ B(x_0, \frac{1}{n}) : 1 \le n \in \mathbb{Z} \}$

$$\begin{array}{ccccccccccc} U_1 & \supset & U_2 & \supset & U_3 & \supset \cdots \supset & U_N & \cdots \supset & U_n & \supset & \cdots \\ \cup & & \cup & & \cup & & & & \cup & & \\ a_1 & & a_2 & & a_3 & \cdots & \cdots & \cdots & a_n & & \end{array}$$

$\forall n \ge N$, $a_n \in U_N$

This is the setting of convergence of a sequence

A sequence in X is a mapping

$$\begin{aligned} \mathbb{N} &\longrightarrow X \\ n &\longmapsto x_n \end{aligned}$$

denoted by $(x_n)_{n=1}^{\infty}$ or $\{x_n\}_{n=1}^{\infty}$

It converges to $x \in X$ if $\forall\, U \in \mathcal{T}$ with $x \in U$
$\exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N$, $a_n \in U$

Also, x is a limit of $(x_n)_{n=1}^{\infty}$; $x_n \to x$

Remark. "$\forall\, U \in \mathcal{T}$ with $x \in U$" can be replaced by
"$\forall\, U \in \mathcal{U}_x$, $\mathcal{U}_x$ a local base at x"

Proposition. Limit of a sequence is unique if the space X is Hausdorff.

Proof. It is very similar to the case of $\mathbb{R}^n$ or metric space. Assume $x_n \to x$, $x_n \to y$ and $x \neq y$.

By Hausdorff, $\exists\, U, V \in \mathcal{T}$, $x \in U$, $y \in V$ and $U \cap V = \emptyset$

By $x_n \to x$, for $U \in \mathcal{T}$, $\exists\, N_1 \in \mathbb{N}$, $\forall\, n \geq N_1$ $a_n \in U$

By $x_n \to y$, for $V \in \mathcal{T}$, $\exists\, N_2 \in \mathbb{N}$, $\forall\, n \geq N_2$, $a_n \in V$

Take $N = \max\{N_1, N_2\}$ Then

$a_N \in U \cap V = \emptyset$ contradiction

Proposition. If a sequence $x_n \in A \longrightarrow x \in X$ then $x \in \bar{A}$

Apparently, there is no analogue of this result in Mathematical Analysis.

Proof. To conclude $x \in \bar{A}$, take any $U \in \mathcal{J}$ and $x \in U$

Then by $x_n \to x$, $\exists$ integer $N \in \mathbb{N}$ such that $\forall\ n \geq N \quad x_n \in U$. In particular $x_N \in U$

Since $x_N \in A$ is given, $x_N \in U \cap A \neq \emptyset$.

Qu. What if A is closed? What implication?

Corollary. Any convergence sequence in a closed set has its limit in the set also.

Now, recall that $x_n \in [a,b]$ converges inside $[a,b]$.

Qu. What about the converse of the proposition? How should we write the converse?

Converse. If $x \in \bar{A}$ then $\exists$ sequence $a_n \in A$, $a_n \to x$.

The converse may not be true in general.

Exercise. Show that the converse is true if X is 1st countable.

Qu. What can be say $x \in A'$? Can it be described by distinct sequence?

Qu How limit of sequence relates to continuity?

$$\lim f(x_n) = f(\lim x_n)$$

Proposition. Let $f: X \longrightarrow Y$ and $x \in X$.
If f is continuous at x then
$\forall$ sequence $x_n \to x$ in X, $f(x_n) \longrightarrow f(x)$ in Y.

Qu. What is the converse?
Is the converse true if $X = \mathbb{R}^n$? Yes
Is the converse true for general X? No

Proof. For $f(x_n) \longrightarrow f(x)$, take any $V \in \mathcal{T}_Y$, $f(x) \in V$
Then $f^{-1}(V) \in \mathcal{T}_X$, $x \in f^{-1}(V)$, as f is continuous
Since $x_n \to x$, $\exists$ $N \in \mathbb{N}$ $\forall$ $n \geq N$ $x_n \in f^{-1}(V)$
Hence $f(x_n) \in V$
It is proved that $f(x_n) \to f(x)$.