Lect07-0202-Convergence

Tuesday, February 2, 2016 9:08 AM

Equivalences of Continuity, $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$

① f is continuous at x, $\forall x\in X$

② $\forall V\in\boxed{\mathcal{T}_Y}$, $f^{-1}(V)\in\boxed{\mathcal{T}_X}$ $\Leftarrow$

③ $\forall V\in\boxed{\mathcal{B}_Y}$, $f^{-1}(V)\in\boxed{\mathcal{T}_X}$

④ $\forall A\subset X$, $f(\overline{A})\subset\overline{f(A)}$

⑤ $\forall B\subset Y$, $\overline{f^{-1}(B)}\subset f^{-1}(\overline{B})$

⑥ $\forall$ closed $H\subset Y$, $f^{-1}(H)$ closed in X

① $\Leftrightarrow$ ② $\Leftrightarrow$ ③ trivial

Need to prove

⑥ $\Rightarrow$ ② $V=Y\setminus H$

$A=f^{-1}(B)$

$H=B=\overline{B}$

② $\Rightarrow$ ④

Let $A\subset X$, want $f(\overline{A})\subset\overline{f(A)}$

i.e. $\forall x\in\overline{A}$ $\underbrace{f(x)\in\overline{f(A)}}$

$\forall V\in\mathcal{T}_Y$ with $f(x)\in V$

$V\cap f(A)\neq\emptyset$

Let $x\in\overline{A}$ and $V\in\mathcal{T}_Y$ with

$f(x)\in V$

Then by ②, $f^{-1}(V)\in\mathcal{T}_X$

Also, $x\in f^{-1}(V)$

So, $f^{-1}(V)$ is a nbhd of $x\in\overline{A}$.

$\therefore$ $f^{-1}(V)\cap A\neq\emptyset$

$\exists$ $a\in f^{-1}(V)\cap A$

$\because$ $f(a)\in V\cap f(A)\neq\emptyset$ Done.

Subspace. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Define

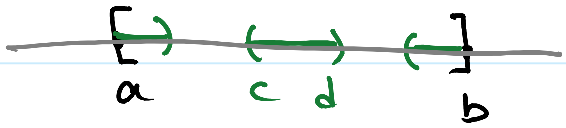
$$\mathcal{T}|_A = \{ G \cap A : G \in \mathcal{T} \}$$

Easy Exercise. $\mathcal{T}|_A$ is a topology of A.

It is called the subspace topology or induced topology or relative topology of A in X

Example. On $(\mathbb{R}, \mathcal{T}_{std})$ and $A = [a, b]$.

Then (c, d) is open in A, $a \leq c < d \leq b$



Also $\underbrace{[a, a+\varepsilon) \text{ and } (b-\varepsilon, b]}_{\text{open in } [a,b] \text{ but not in } \mathbb{R}}$ are in $\mathcal{T}|_A$

Qu. Can a continuous function on a subspace determine a continuous function on X?

Let us deal with the question in easy cases.

Proposition. Let $(X, \mathcal{T})$ be a topological space; $f : X \longrightarrow Y$ and $X = \bigcup_{\alpha \in I} G_\alpha$, each $G_\alpha \in \mathcal{T}$.

If each $f_\alpha = f|_{G_\alpha} : G_\alpha \longrightarrow Y$ is continuous on the subspace G_α of X

then f is continuous on X.

Monday, February 15, 2016 4:51 PM

The essence of the result: continuity on open subspaces forming the space will guarantee continuity on the whole

Equivalent Proposition Let $(X, \mathcal{J})$ and G_α be as before having the subspace topology. If $f_\alpha : G_\alpha \longrightarrow Y$ are continuous mappings such that $f_\alpha = f_\beta$ on $G_\alpha \cap G_\beta$ then $\exists\ f : X \longrightarrow Y$ which is continuous and $f|_{G_\alpha} = f_\alpha$ for each α.

Clearly, for this version, we simply define

$$f(x) = f_\alpha(x) \quad \text{if} \quad x \in G_\alpha$$

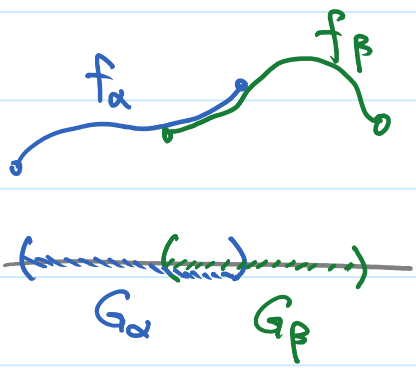
Then it is well-defined for $x \in G_\alpha \cap G_\beta$.

Main Idea: Let $V \in \mathcal{J}_Y$

Then $f^{-1}(V)$

$$\bigcup_{\alpha \in I} \left[f^{-1}(V) \cap G_\alpha \right]$$

$$= \bigcup_{\alpha \in I} f_\alpha^{-1}(G_\alpha) \quad \text{each in } \mathcal{J}$$



From this, one sees that the situation for closed subspaces is different.

Proposition. Let $(X, \mathcal{T})$ be topological space and $A, B \subset X$ be closed; $f: X \longrightarrow Y$.
If $f|_A : A \longrightarrow Y$ and $f|_B : B \longrightarrow Y$ are continuous under the subspace topologies then f is continuous.

Exercise. (a) Formulate an equivalent version
(b) Prove it.
(c) Give an example of closed A_α such that $f|_{A_\alpha}$ are continuous but not f.

Previous cases, we know continuity of $f|_{\text{subspace}}$ and they form the whole X.

Qu Give an example of $f: \mathbb{R} \longrightarrow \mathbb{R}$ such that $f|_{\mathbb{Q}} \equiv 0$.

(1) Dirichlet Function

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

(2) A continuous function ??

$$f(x) \equiv 0 \quad \forall\ x \in \mathbb{Q}$$

Qu Is there another continuous example?

The answer is no. Why?

Now, f is determined but only $f|_{\mathbb{Q}}$ is known.

Uniqueness Theorem. Let $A \subset X$ be dense; Y be Hausdorff and $f, g: X \longrightarrow Y$ be continuous.

If $f|_A \equiv g|_A$ then $f \equiv g$ on X.

Qu. Can the above theorem used to answer the following question: find a continuous $f: \mathbb{R} \longrightarrow \mathbb{R}$ with $f(\frac{p}{q}) = \frac{1}{q}$ $\forall \frac{p}{q} \in \mathbb{Q}$.

No, the theorem only tells us whether two functions are the same but not whether any of them exists.

Proof of the theorem

Need to prove $\underbrace{f(x) = g(x)}_{\text{in } Y \text{, which is Hausdorff}}$ for arbitrary $x \in X$

Hausdorff is usually easier to write for $y_1 \neq y_2$, so we expect proving by contradiction.

Suppose $\exists$ $x \in X$, $f(x) \neq g(x)$

Then $\exists$ $V_1, V_2 \in \mathcal{T}_Y$, $V_1 \cap V_2 = \phi$

$$f(x) \in V_1 , \quad g(x) \in V_2$$

By continuity, $x \in f^{-1}(V_1) \cap g^{-1}(V_2) \in \mathcal{T}_X$

By density of A, $\exists$ $a \in f^{-1}(V_1) \cap g^{-1}(V_2) \cap A$

$$f(a) = g(a) \in V_1 \cap V_2 \quad \text{contradiction}$$

Definition. A mapping $f: X \longrightarrow Y$ is called a homeomorphism if it is a bijection and both f and f^{-1} are continuous. In this case, $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are having the same topological structure.

An open mapping $f: X \longrightarrow Y$ satisfies that $\forall\ U \in \mathcal{T}_X,\ f(U) \in \mathcal{T}_Y$

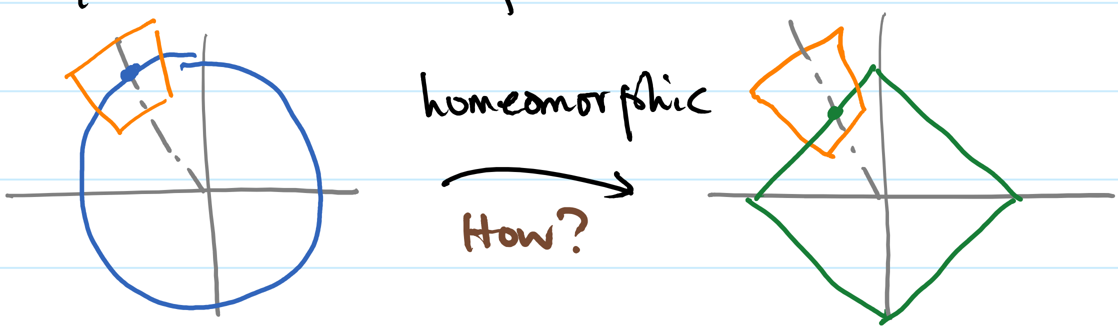
Exercise. Find an example for each case

* continuous but not open
* open but not continuous
* open and continuous but not homeomorphism

Example Euclidean topology of $\mathbb{R}^2$

$X = S^1$
$= \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$

$Y = \{y \in \mathbb{R}^2 : |y_1| + |y_2| = 1\}$



$f(x) = y$ if $\begin{matrix} x \in S^1 \\ y \in Y \end{matrix}$ both lie on the same radial line

To show continuity (or open), take suitable open set $V \cap Y$ and consider $f^{-1}(V) \cap X$.