Pf of Thm 10: It is clear that we only need to show the cases of $K = 0, +1, -1$. And we may assume $M = \mathbb{R}^n, S^n$ or $\mathbb{H}^n$.

Case 1: $K = 0$ or -1.

Since $K \le 0$, Cartan Hadamard $\Rightarrow$

$$\left\{\begin{array}{l} \exp_x^M : T_xM \to M \\ \\ \exp_y^N : T_yN \to N \end{array}\right. \quad \text{are diffeomorphisms.}$$

Let $\Phi : T_xM \to T_yN$ be the unique isometry between the inner product spaces T_xM & T_yN s.t.

$$\Phi(e_i) = \varepsilon_i \quad \forall\, i = 1, \cdots, n.$$

Define $\varphi : M \to N$ by

$$\varphi = \exp_y^N \circ \Phi \circ (\exp_x^M)^{-1}$$

$$\begin{tikzcd} T_xM \arrow[r, "\Phi"] \arrow[d, "\exp_x^M"'] & T_yN \arrow[d, "\exp_y^N"] \\ M \arrow[r] & N \end{tikzcd}$$

Clearly φ is a diffeomorphism. We need to show that φ is an isometry.

i.e. $\forall\, z \in M$ and $X \in T_zM$, we have

$$|d\varphi(X)|_N = |X|_M.$$

By Cartan-Hadamard,

$$\exists\ T \in T_x M \quad \text{and} \quad w \in T_T(T_x M) \cong T_x M \text{ s.t.}$$

$$z = \exp_x^M(T) \quad \text{and} \quad \xi = (d\exp_x^M)_T(w).$$

By the identification $T_T(T_xM) \cong T_xM$, we can define a 1-parameter family of geodesics

$$\gamma_u(t) = \exp_x^M\left[t(T+uw)\right].$$

Let $U(t)$ = transversal vector field of γ_u along γ_0.

Then (from the fact (B) in the proof of Cartan-

Hadamard), $U(t)$ is a Jacobi field s.t.

$$\begin{cases} U(0) = 0 \\ U'(0) = w \end{cases}$$

and further $U(1) = (d\exp_x^M)_T(w) = X$.

In N, we define correspondingly

$$\gamma_u^N(t) = \exp_y^N\left[t\left(\Phi(T) + u\,\Phi(w)\right)\right]$$

& $U^N(t)$ = transversal vector field of $\{\gamma_u^N\}$ along γ_0^N.

Then U^N is a Jacobi field along $\gamma_0^N \subset N$

s.t.

$$\begin{cases} \mho^N(0) = 0 \\ (\mho^N)'(0) = \Phi(w). \end{cases}$$

Note that

$$\varphi(\gamma_u(t)) = \exp_y^N \circ \Phi \circ (\exp_x^M)^{-1}(\exp_x^M[t(T+uw)])$$

$$= \exp_y^N \circ \Phi(t(T+uw))$$

$$= \exp_y^N[t(\Phi(T)+u\Phi(w))]$$

$$= \gamma_u^N(t).$$

$$\Rightarrow \quad d\varphi(\mho(t)) = \mho^N(t) \quad \text{(by differentiation)}$$

$$\Rightarrow \quad \mho^N(1) = d\varphi(\mho(1)) = d\varphi(X).$$

Therefore, what we need to show is

$$|V^N(1)| = |V(1)|.$$

To see this, we use parallel orthonormal frames $\{e_1(t), \cdots, e_n(t)\}$ & $\{\varepsilon_1(t), \cdots, \varepsilon_n(t)\}$ along γ_0 and γ_0^N respectively s.t.

$$\begin{cases} e_i(0) = e_i \\ \varepsilon_i(0) = \varepsilon_i \end{cases} \quad \forall\, i = 1, \cdots, n.$$

Then

$$\begin{cases} V(t) = \sum_i f_i(t) e_i(t) \\ V^N(t) = \sum_i g_i(t)\, \varepsilon_i(t) \end{cases}$$

for some functions $f_i(t)$ & $g_i(t)$.

Let $V_0(t) = \dfrac{\gamma_0'(t)}{|\gamma_0'(t)|}$, then

$$R_{\gamma_0'(t)U(t)}\gamma_0'(t) = |\gamma_0'(t)|^2 R_{V_0(t)U(t)}V_0(t)$$

$$(\text{Lemma 12}) \quad = |\gamma_0'(t)|^2 K\left[U(t) - \langle U(t), V_0(t)\rangle V_0(t)\right]$$

Since $\langle \gamma_0'(t), \gamma_0'(t)\rangle = \langle \gamma_0'(0), \gamma_0'(0)\rangle = |T|^2$

$$\langle \gamma_0'(t), e_i(t)\rangle = \langle T, e_i\rangle,$$

we have

$$U''(t) + R_{\gamma_0'(t)U(t)}\gamma_0'(t) = 0$$

$$\Longleftrightarrow \quad \sum f_i'' e_i + |r_0'|^2 K \left[\sum f_i e_i - \frac{\langle \sum f_i e_i , r_0' \rangle r_0'}{|r_0'|^2} \right] = 0$$

$$\Longleftrightarrow \quad \sum \left(f_i'' + |T|^2 K f_i \right) e_i - K \sum_i f_i \langle e_i , r_0' \rangle r_0' = 0$$

$$\Longleftrightarrow \quad \sum \left(f_i'' + |T|^2 K f_i \right) e_i - K \sum_i f_i \langle e_i , T \rangle \sum_j \langle e_j , r_0' \rangle e_j = 0$$

$$\Longleftrightarrow \quad \sum_i \left(f_i'' + |T|^2 K f_i \right) e_i - K \sum_{i,j} f_i \langle e_i , T \rangle \langle e_j , T \rangle e_j = 0$$

$$\Longleftrightarrow \quad \sum_i \left[f_i'' + |T|^2 K f_i - K \sum_j f_j \langle e_j , T \rangle \langle e_i , T \rangle \right] e_i = 0$$

$$\Longleftrightarrow \quad f_i'' + \sum_j f_j K \left[|T|^2 \delta_{ij} - \langle T, e_i \rangle \langle T, e_j \rangle \right] = 0 , \quad \forall i = 1, \ldots, n.$$

Furthermore $V(0)=0$ & $V'(0)=w \Rightarrow$

$$\begin{cases} f_i(0)=0 \\ f_i'(0)=\langle w, e_i\rangle \end{cases}$$

$\therefore$

$$\begin{cases} f_i'' + \sum_j f_j K\left[|T|^2\delta_{ij} - \langle T, e_i\rangle\langle T, e_j\rangle\right] = 0, \\ f_i(0)=0 \\ f_i'(0)=\langle w, e_i\rangle \end{cases}$$

Similarly, we have

$$\begin{cases} g_i'' + \sum_j g_j K\left[|\Phi(T)|^2\delta_{ij} - \langle \Phi(T), \varepsilon_i\rangle\langle \Phi(T), \varepsilon_j\rangle\right] = 0, \\ g_i(0)=0 \\ g_i'(0)=\langle \Phi(w), \varepsilon_i\rangle \end{cases}$$

Using the fact Φ is an isometry (between inner product spaces T_xM & T_yN) we have

$$\begin{cases} |\Phi(T)|^2 = |T|^2 \\ \langle \Phi(T), \varepsilon_i \rangle = \langle \Phi(T), \Phi(e_i) \rangle = \langle T, e_i \rangle \\ \langle \Phi(w), \varepsilon_i \rangle = \langle w, e_i \rangle . \end{cases}$$

$\therefore$ $\{f_i\}$ & $\{g_i\}$ satisfy the same IVP of an ODE system.

$$\Rightarrow \quad f_i \equiv g_i , \quad \forall t , \ \forall i = 1, \cdots, n .$$

Therefore $$|\sigma^N(1)|^2 = \sum_i g_i(1)^2 = \sum_i f_i(1)^2 = |\sigma(1)|^2 .$$

This completes the proof of the case that $K = 0$ or -1.

Case of $K = +1$

We may assume $M = S^n$.

If $\bar{x} = -x$ (the antipodal point of x), then

$$(\exp_x^M)^{-1} : S^n \setminus \{\bar{x}\} \to T_x S^n \text{ is well-defined.}$$

Therefore, we can define similarly the map

$$\varphi = \exp_y^N \circ \Phi \circ (\exp_x^M)^{-1} : S^n \setminus \{\bar{x}\} \to N.$$

Similar argument shows that φ is a <u>local</u> isometry (not necessary global yet.)

Observe that $\forall\, z \in S^n \setminus \{x, \bar{x}\}$, we still have

$$\begin{tikzcd} T_z S^n \arrow[r, "d\varphi"] & T_{\varphi(z)} N \arrow[d, "\exp^N_{\varphi(z)}"] \\ S^n \setminus \{\bar{x}, \bar{z}\} \arrow[u, "(\exp^M_z)^{-1}"] \arrow[r, "\varphi"] \arrow[ur, phantom, "\circlearrowright"] & N \end{tikzcd}$$

as φ is a local isometry.

Note that $d\varphi|_{T_z S^n} : T_z S^n \to T_{\varphi(z)} N$ is an inner product space isometry, same argument above implies that

$$\psi : S^n \setminus \{\bar{z}\} \to N$$

defined by $\psi = \exp^N_{\varphi(z)} \circ d\varphi\big|_{T_z S^n} \circ (\exp^{S^n}_z)^{-1}$

is a local isometry. By the above commutative

diagram, $\forall\, p \in S^n \setminus \{\bar{x}, \bar{z}\}$

$$\varphi(p) = (\exp^N_{\varphi(z)}) \circ d\varphi \circ (\exp^{S^n}_z)^{-1}(p)$$

$$= (\exp^N_{\varphi(z)}) \circ d\varphi\big|_{T_z S^n} \circ (\exp^{S^n}_z)^{-1}(p)$$

$$= \psi(p)$$

Therefore, we can extend φ to define on S^n by setting

$$\varphi(\bar{x}) = \psi(\bar{x}).$$

Then by construction $\varphi : S^n \to N$ is a local isometry.

Similarly argument as in

~~Hence~~ Lemma 8 $\Rightarrow$ φ is a covering map. Since N is simply-connected, φ has to be an isometry.

It is clear from the construction, $d\varphi(e_i) = \varepsilon_i$, $\forall i = 1, \cdots, n$.

So we've proved the existence part of Thm 10.

Finally, the uniqueness follows from:

<u>Lemma 13</u>: Let $\varphi_i : M \to N$, $i = 1, 2$ be 2 local isometries

between complete Riem. mfds M & N s.t. for some $x \in M$, $\varphi_1(x) = \varphi_2(x)$ (in N) and

$$d\varphi_1 \big|_{T_xM} = d\varphi_2 \big|_{T_xM}.$$

Then $\varphi_1 \equiv \varphi_2$.

<u>Pf</u>: Let $S = \{ z \in M : \varphi_1(z) = \varphi_2(z) \text{ \& } d\varphi_1 \big|_{T_zM} = d\varphi_2 \big|_{T_zM} \}$.

- By assumption, $x \in S$. $\therefore S \neq \emptyset$.
- It is clearly that S is closed by continuity.
- If $z \in S$, take $\delta > 0$ s.t.

  $\exp_z^M : B(\delta) \to M$ is a diffeo. injection, (where $B(\delta) = \{ X \in T_z M : |X| < \delta \}$).

Recall that we have commutative diagram

$$\begin{tikzcd} T_z M \arrow[r, "d\varphi"] \arrow[d, "\exp_z^M"'] & T_{\varphi(z)} N \arrow[d, "\exp_{\varphi(z)}^N"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

$\forall$ local isometry φ.

Applying this to φ_1 & φ_2, we have

$$\exp_z^M (B(\delta)) \subset S. \qquad \text{(Ex!)}$$

$\Rightarrow S$ is open.

Therefore, by connectedness of $M \Rightarrow S = M$ $\#$

This complete the proof of Thm 10. $\#$

Cor 14: Let M = complete simply-connected Riem. mfd of dim $= n$.

Then M is a space form

$\Leftrightarrow$ $\forall x, y \in M$ and

$\forall$ orthonormal bases $\{e_i\}$ of T_xM &

$\{\varepsilon_i\}$ of T_yM,

$\exists$ isometry $\varphi : M \to M$ s.t. $\varphi(x) = y$ and

$d\varphi(e_i) = \varepsilon_i \quad \forall i$.

(Pf: Immediately from Thm 10)

Note: Cor 14 proves that simply-connected space form is homogeneous. In fact, we have more

Cor 15: simply-connected space forms are two-points homogeneous.

Def: M is called two-points homogeneous if $\forall\, p_1, p_2, q_1, q_2 \in M$ with $d(p_1, p_2) = d(q_1, q_2)$, $\exists$ an isometry $\varphi : M \to M$ s.t.

$$\varphi(p_1) = q_1 \quad \& \quad \varphi(p_2) = q_2 .$$

Pf of Cor 15:

Let p_1, p_2, q_1, q_2 be points in a simply-connected space form M s.t. $d(p_1, p_2) = d(q_1, q_2) = \alpha$.

Let $\zeta, \xi : [0, \alpha] \to M$ be normalized geodesics s.t.

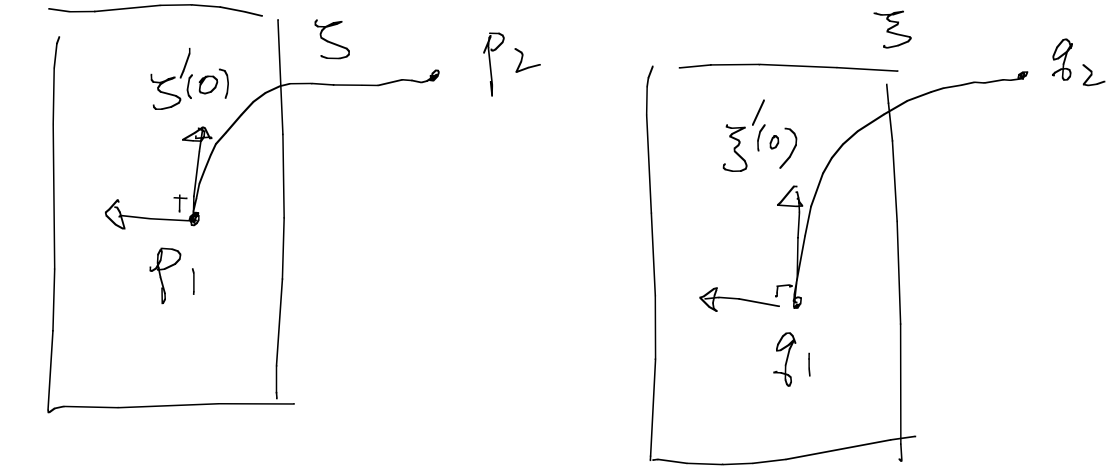
$$\zeta(0) = p_1, \quad \zeta(\alpha) = p_2$$

$$\xi(0) = q_1, \quad \xi(\alpha) = q_2$$

Choose orthonormal bases

$\{e_i\}$ on $T_{p_1}M$ s.t. $e_1 = \zeta'(0)$ &

$\{\varepsilon_i\}$ on $T_{q_1}M$ s.t. $\varepsilon_1 = \xi'(0)$



Then Thm 10 (a (or 14) $\Rightarrow$ $\exists$ isometry $\varphi : M \to M$

s.t. $\varphi(p_1) = q_1$ & $d\varphi(e_i) = \varepsilon_i$

$\Rightarrow$ $\varphi \circ \zeta$ & ξ are geodesics with same initial data, hence $\varphi \circ \zeta = \xi$.

$\Rightarrow$ $\varphi(p_2) = q_2$. ⌘

Ch 7 The 1st & 2nd variation famula

Let
- M = complete Riem. mfd
- $\gamma(t,u) : [a,b] \times [-\varepsilon, \varepsilon] \to M$ a C^∞ map,
- $\{\gamma_u(t)\}$ corresponding 1-parameter family of curves with base curve γ_0 equal to a given curve $\gamma(t)$ parametrized by arc-length, i.e. $|\gamma'(t)| = 1$.
- U = transversal vector field of $\{\gamma_u\}$
- T = tangent vector field along $\{\gamma_u\}$.

Then the length of $\gamma_u(t)$ is

$$L(u) = \int_a^b |\gamma_u'(t)|\,dt = \int_a^b |T|\,dt$$

$$\therefore \quad \frac{dL}{du}(u) = \int_a^b \frac{d}{du}|T|\,dt$$

$$= \int_a^b U\sqrt{\langle T, T\rangle}\,dt$$

$$= \int_a^b \frac{\langle T, D_U T\rangle}{|T|}\,dt$$

$$= \int_a^b \frac{1}{|T|}\langle T, D_T U\rangle\,dt \qquad \text{since } [T, U] = 0$$

Putting $u=0$,

$$\frac{dL}{du}(0) = \int_a^b \langle \gamma'(t), D_{\gamma'(t)} U \rangle \, dt$$

$$= \int_a^b \left[\frac{d}{dt} \langle \gamma'(t), U \rangle - \langle D_{\gamma'(t)} \gamma'(t), U \rangle \right] dt$$

where $U(t) = U(t,0)$ is the transversal vector field along γ.

$$\Rightarrow \boxed{\frac{dL}{du}(0) = \langle \gamma'(t), U(t) \rangle \Big|_a^b - \int_a^b \langle D_{\gamma'(t)} \gamma'(t), U(t) \rangle \, dt}$$

which is the 1^{st} variation formula for arc-length.

Lemma 1: A curve $\gamma : [a,b] \to M$ is a geodesic if and only if it is a critical point of the arc-length functional with respect to $\overbrace{\text{normal variations}}^{(\text{all})}$ $\{\gamma_u\}$ (i.e. $\forall u$, $\gamma_u(a)=\gamma(a)$ & $\gamma_u(b)=\gamma(b)$.)

Pf: For normal variations, $V(a)=V(b)=0$

$$\therefore \quad \frac{dL}{du}(0) = -\int_a^b \langle D_{\gamma'}\gamma', V\rangle dt$$

$\forall V$ with $V(a)=V(b)=0$.

$$\therefore \quad 0 = \underset{\substack{\forall V \\ \text{with } V(a)=V(b)=0}}{\frac{dL}{du}(0)} \iff D_{\gamma'}\gamma' = 0 \qquad (\text{Ex!})$$

#

Lemma 2: Let
- N = closed submanifold of M
- $x \notin N$
- $y \in N$ s.t.

$d(x,y) = d(x,N) \overset{\text{def}}{=} \inf\{d(x,y) : y \in N\}$

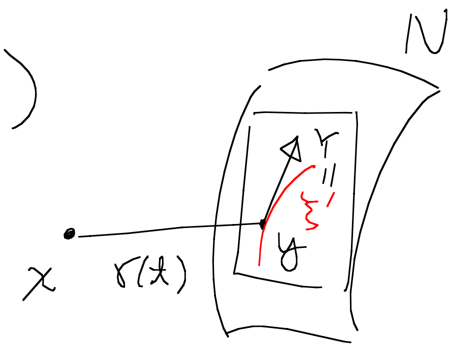
- $\gamma : [a,b] \to M$ shortest geodesic joining x to y.

Then γ is normal to N (i.e. $\gamma'(b) \perp T_yN$).

Pf: Let $Y \in T_yN$. We need to show that $\langle \gamma'(b), Y\rangle = 0$.

For this, take a C^∞ curve $\xi : [-\varepsilon, \varepsilon] \to N$ s.t.

$$\xi'(0) = Y \quad (\xi(0) = y)$$



Let $\{\gamma_u\}$ be a 1-parameter family of curves given by

$\gamma(t,u) : [a,b]\times[-\varepsilon,\varepsilon] \longrightarrow M$ with

$$\begin{cases} \gamma_0(t) = \gamma(t) , & \forall t\in[a,b] \\ \gamma_u(a) = x , & \forall u \\ \gamma_u(b) = \xi(u) . \end{cases}$$

By assumption

$$L(0) = d(x,y) \le d(x,\xi(u)) \le L(u) , \quad \forall u\in[-\varepsilon,\varepsilon]$$

$$\Longrightarrow \quad \frac{dL}{du}(0) = 0 .$$

1^{st} variation formula $\Longrightarrow$

$$0 = \langle \gamma'(t), V(t)\rangle\Big|_a^b - \int_a^b \langle \cancel{D_{\gamma'}}\gamma', V\rangle dt$$

$$= \langle \gamma'(b), V(b)\rangle - \langle \gamma'(a), V(a)\rangle \quad \left(\text{since } \gamma \text{ is a geodesic}\right)$$

Note that $U(a) = 0$ &

$$U(b) = \frac{\partial}{\partial u}\Big|_{u=0} \gamma_u(b) = \frac{d}{du}\Big|_{u=0} \xi(u) = \xi'(0) = Y.$$

$$\therefore \quad \langle \gamma'(b), Y \rangle = 0 \qquad \#$$

Now suppose that $\gamma : [a,b] \to M$ is a <u>normalized geodesic</u>.

We would like to calculate $\frac{d^2 L}{du^2}(0)$ for the family $\{\gamma_u\}$.

We've proved that

$$\frac{dL}{du}(u) = \int_a^b \frac{1}{|T|} \langle T, D_T U \rangle \, dt$$

$$\Rightarrow \quad \frac{d^2L}{du^2}(u) = \int_a^b \frac{d}{du}\left[\frac{1}{|T|}\langle T, D_T U\rangle\right] dt$$

$$= \int_a^b \left\{ -\frac{1}{|T|^3}\langle T, D_T U\rangle^2 + \frac{1}{|T|} U\langle T, D_T U\rangle \right\} dt$$

$$= \int_a^b \left\{ \begin{array}{l} -\frac{1}{|T|^3}\langle T, D_T U\rangle^2 \\ + \frac{1}{|T|}\langle D_U T, D_T U\rangle + \frac{1}{|T|}\langle T, D_U D_T U\rangle \end{array} \right\} dt$$

$$= \int_a^b \left\{ \begin{array}{l} -\frac{1}{|T|^3}\langle T, D_T U\rangle^2 + \frac{1}{|T|}|D_T U|^2 \\ + \frac{1}{|T|}\langle T, D_T D_U U + R_{TU} U\rangle \end{array} \right\} dt$$

(since $[U, T] = 0$)

$$= \int_a^b \left\{ -\frac{1}{|T|^3}\left[T\langle T, U\rangle - \langle D_T T, U\rangle\right]^2 + \frac{1}{|T|}|D_T U|^2 + \frac{1}{|T|}\langle T, D_T D_U U\rangle - \frac{1}{|T|}\langle R_{UT}U, T\rangle \right\} dt$$

Note that at $u=0$, $D_T T = D_{\gamma'}\gamma' = 0$

$$|T| = |\gamma'| = 1.$$

$$\Rightarrow \frac{d^2L}{du^2}(0) = \int_a^b \left\{ -\left[\frac{d}{dt}\langle \gamma', U\rangle\right]^2 + |U'|^2 + \langle \gamma', D_{\gamma'} D_U U\rangle - \langle R_{U\gamma'}U, \gamma'\rangle \right\} dt$$

where $U' = D_{\gamma'}U$.

$$\Rightarrow \frac{d^2L}{du^2}(0) = \int_a^b \left\{ -\left[\frac{d}{dt}\langle \gamma', U\rangle\right]^2 + \frac{d}{dt}\langle \gamma', D_U U\rangle + |U'|^2 - \langle R_{U\gamma'}U, \gamma'\rangle \right\} dt$$

$\Rightarrow$

$$\boxed{\frac{d^2L}{du^2}(0) = \langle \gamma', D_{\gamma'} V\rangle\Big|_a^b + \int_a^b \left\{ \left(|V'|^2 - \left[\frac{d}{dt}\langle \gamma', V\rangle\right]^2 \right) - \langle R_{V\gamma'}V, \gamma'\rangle \right\} dt}$$

which is the 2nd variation formula.

Let $V^\perp = V - \langle V, \gamma'\rangle \gamma'$ the normal component of V, then the 2nd variation formula can be written as

$$\boxed{\frac{d^2L}{du^2}(0) = \langle \gamma', D_{V} V\rangle\Big|_a^b + \int_a^b \left\{ |D_{\gamma'} V^\perp|^2 - \langle R_{V^\perp \gamma'} V^\perp, \gamma'\rangle \right\} dt}$$

(Ex!)

Note: • If $\{\gamma_u\}$ is normal in the sense that

$$\gamma_u(a) = \gamma(a), \quad \gamma_u(b) = \gamma(b),$$

then $\langle \gamma', D_V V\rangle(a) = \langle \gamma', D_V V\rangle(b) = 0$.

• If $\{\gamma_u\}$ is a 1-parameter of (smooth) closed curves,

then $\langle \gamma', D_V V\rangle\Big|_a^b = 0$

• The interior term $\displaystyle\int_a^b \left[|D_{\gamma'} V^\perp|^2 - \langle R_{V^\perp, \gamma'} V^\perp, \gamma'\rangle\right] dt$

$$= \int_a^b \left[|D_{\gamma'} V^\perp|^2 - \langle R_{\gamma' V^\perp}\gamma', V^\perp\rangle\right] dt$$

is related to the Jacobi Operator on $V^\perp$ (provided a right bdy condition)

(Ex!)