Department of Mathematics
The Chinese University of Hong Kong

MAT5061 Riemannian Geometry I
Final Examination

Apr 28, 2014

Answer all questions and show all your steps.

(1) (30 marks) Show the Ricci Identity

$$D^2T(\cdots, X, Y) - D^2T(\cdots, Y, X) = (R_{XY}T)(\cdots)$$

for any smooth tensor field T, where R is the curvature tensor.

(2) (30 marks) Let $\varphi : N \to M$ be a C^∞ map from a C^∞ manifold N to a Riemannian manifold M with Levi-Civita Connection D. Define the induced connection $\tilde{D}$ of D along the map φ by writing down a formula for $\tilde{D}_V X$, where V is a smooth vector field on N and X is a smooth vector field along φ. Then show that for any smooth vector fields V and W on N,

$$\tilde{D}_V d\varphi(W) - \tilde{D}_W d\varphi(V) - d\varphi([V, W]) = 0.$$

(3) (40 marks) Let M be a complete Riemannian manifold and $\gamma : [0.l] \to M$ a geodesic parametrized by arc-length. Suppose that there is no conjugate point of $\gamma(0)$ along γ and that there is a real number β such that for any 2-plane $\Pi \subset T_{\gamma(t)}M$, $t \in [0, l]$, containing $\gamma'(t)$, the sectional curvature $K(\Pi) \leq \beta$. Show that for any normal Jacobi field $U(t)$ along γ with $U(0) = 0$ and $\lim_{t\to 0}\langle \dot{U}, \frac{U}{|U|}\rangle = 1$,

$$|U(t)| \geq \begin{cases} \frac{1}{\sqrt{\beta}} \sin \sqrt{\beta} t, & \beta > 0 \text{ and } t \in [0, \frac{\pi}{2\sqrt{\beta}}), \\ t, & \beta = 0, \\ \frac{1}{\sqrt{-\beta}} \sinh \sqrt{-\beta} t, & \beta < 0. \end{cases}$$

(End)