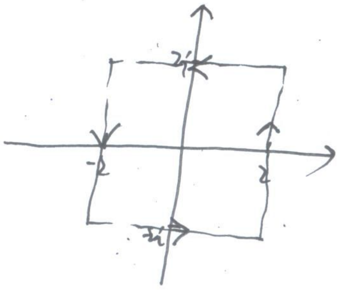
MATH 2230 Tutorial 8

Cauchy integral formula: $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\, ds$

- $f^{n}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}\, ds$

1. Compute: a) $\int_C \frac{\cos z}{z(z^2+8)}\, dz$

b) $\int_C \frac{\cosh z}{z^4}\, dz$.



Ans: a) $\because \frac{\cos z}{z(z^2+8)}$ is analytic inside C except 0.

$\therefore$ Let $f(z) = \frac{\cos z}{z^2+8}$

then $f(z)$ is analytic inside C

$\therefore$ By Cauchy integral formula

$$\frac{1}{8} = f(0) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-0}\, ds = \frac{1}{2\pi i}\int_C \frac{\cos s}{s\cdot(s^2+8)}\, ds$$

$$\therefore \int_C \frac{\cos z}{z(z^2+8)}\, dz = \frac{\pi i}{4}$$

b) $\because \frac{\cosh z}{z^4}$ is analytic inside C except 0

$\therefore$ Let $g(z) = \cosh z$

then $g(z)$ is analytic inside C

$\therefore$ We can apply the Cauchy integral formula

$$g^{(3)}(0) = \frac{3!}{2\pi i}\int_C \frac{g(z)}{(z-0)^4}\, dz = \frac{3}{\pi i}\int_C \frac{\cosh z}{z^4}\, dz$$

$\because g^{3}(0) = \sinh z \big|_0 = 0$

$$\therefore \int_C \frac{\cosh z}{z^4}\, dz = 0$$

2. Let $f(z)$ be an entire function on the complex plane

Suppose $|f(z)| \leq A|z|^k$ for some positive constant A
positive integer k
when $|z|$ is sufficient large.

then $f(z)$ is a polynomial of order less than or equal to

Ans: $\because f(z)$ is an entire function

then $\forall R>0$, let $\gamma = \{Re^{i\theta} \mid \theta \in [0, 2\pi)\}$

By Cauchy integral formula

$$f^{n}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^{n+1}} ds \quad \text{when } z \text{ is inside } \gamma$$

$$\therefore |f^{n}(z)| \leq \frac{n!}{2\pi} \int_\gamma \frac{|f(s)|}{|s-z|^{n+1}} |ds|$$

$$\leq \frac{n!}{2\pi} \int_\gamma \frac{A|s|^k}{(|s|-|z|)^{n+1}} |ds|$$

$$= \frac{n!}{2\pi} \int_\gamma \frac{AR^k}{(R-|z|)^{n+1}} |ds|$$

$$= \frac{n!}{2\pi} \cdot \frac{AR^k}{(R-|z|)^{n+1}} \cdot 2\pi R$$

$$= \frac{n!}{1} \cdot \frac{A}{(R-|z|)^{n+1}} \cdot R^{k+1} \quad \text{for any } R>|z|.$$

$\therefore$ For $n>k$

let $R \to \infty$, then $|f^{n}(z)| \leq 0$

$\therefore f^{n}(z) = 0$

$\therefore f^{n}(z) = 0$ for $\forall n>k$, $\forall z \in \mathbb{C}$

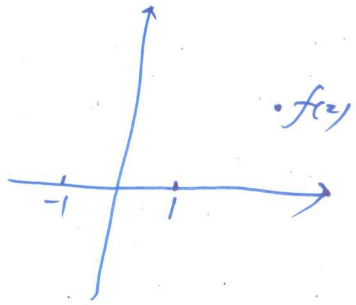
which means $f(z)$ is a polynomial of order less than or equal to k.

3. Let $f(z)$ be an entire function on $\mathbb{C}$.

Let u be the real part of f

Suppose $u \geq 1$

Show that $f(z)$ is a constant.



Ans: Let $h(z) = \dfrac{z-1}{z+1}$

Let $g(z) = h \circ f(z) = \dfrac{f(z)-1}{f(z)+1}$

$\because$ $Re(f) = u \geq 1$

$\therefore$ $f(z)+1 \neq$ for $\forall z$

$\therefore$ $g(z)$ is an entire function

$\because$ $Re(f) = u \geq 1$

$\therefore$ $|f(z)-1| \leq |f(z)+1|$ (check!)

$\therefore$ $|g(z)| \leq 1 = 1 \cdot |z|^0$

$\therefore$ By Q2, $g(z)$ is a polynomial of order less than or equal to 0

which means $g(z)$ is a constant

$\therefore$ $f(z)$ is a constant

- The result still holds if
$$\begin{cases} Re(f) \geq C_0 \\ Re(f) \leq C_1 \\ Im(f) \geq C_2 \\ Im(f) \leq C_3 \end{cases}$$

We still have a theorem which says

if f is entire, and $Re(f) \le A|z|^k$ when $|z|$ is sufficient large.
then $f(z)$ is a polynomial of order less than or equal to k.

Pf: Let $u = Re(f)$

$\therefore$ By assumption

For large $|z|$, $|u| \le A|z|^k$

Fix z, let $\gamma = \{ z + Re^{i\theta} : \theta \in [0, 2\pi) \}$

By Cauchy integral formula

$$f^n(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^{n+1}} ds$$

$$= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z+Re^{i\theta})}{(Re^{i\theta})^{n+1}} iRe^{i\theta} d\theta$$

$$= \frac{n!}{2\pi R^n} \int_0^{2\pi} f(z+Re^{i\theta}) e^{-in\theta} d\theta$$

when $n \ge 0$, $f(s) \cdot (s-z)^n$ is entire

$$\therefore \int_\gamma f(s)(s-z)^n ds = 0$$

$$\therefore \int_0^{2\pi} f(z+Re^{i\theta}) R^n e^{in\theta} iRe^{i\theta} d\theta = 0$$

$$\therefore \int_0^{2\pi} f(z+Re^{i\theta}) e^{i(n+1)\theta} d\theta = 0$$

$$\therefore \overline{\int_0^{2\pi} f(z+Re^{i\theta}) e^{i(n+1)\theta} d\theta} = 0$$

$$\therefore \int_0^{2\pi} \overline{f(z+Re^{i\theta})}\, e^{-i(n+1)\theta} d\theta = 0$$

$\because f^0(z) = f^0(z)$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}+z) d\theta.$$

$$\therefore Re(f(z)) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}+z) d\theta$$

$$\therefore |f^n(z)| \le \frac{2n!}{R^n} A(R+|z|)^k - \frac{2n!}{R^n} Re(f(z))$$

$\to 0$ as $R \to \infty$ if $n > k$.

$\therefore f^n(z) \equiv 0$ for $n > k$

$\therefore f$ is a polynomial of order less than or equal to k.

$\because 2u = f + \bar{f}$

$$f^n(z) = \frac{n!}{2\pi R^n} \int_0^{2\pi} f(z+Re^{i\theta}) e^{-in\theta} d\theta + \underbrace{\frac{n!}{2\pi R^n} \int_0^{2\pi} \overline{f(z+Re^{i\theta})}\, e^{-in\theta} d\theta}_{=0} \quad \text{for } n > k$$

$$= \frac{n!}{2\pi R^n} \int_0^{2\pi} 2u e^{-in\theta} = \frac{n!}{\pi R^n} \int_0^{2\pi} (u - A|s|^k) e^{-in\theta} d\theta$$

since $\int_0^{2\pi} A|z|^k e^{-in\theta} d\theta = 0$ for $n > k \ge 0$

$$\therefore |f^n(z)| \le \frac{n!}{\pi R^n} \int_0^{2\pi} |A|s|^k - u| d\theta = \frac{n!}{\pi R^n} \int_0^{2\pi} (A|s|^k - u) d\theta$$

since $A|s|^k - u \ge 0$ where $s = Re^{i\theta} + z$

$$\le \frac{2n!}{R^n} A(R+|z|)^k - \frac{n!}{\pi R^n} \int_0^{2\pi} u\, d\theta$$