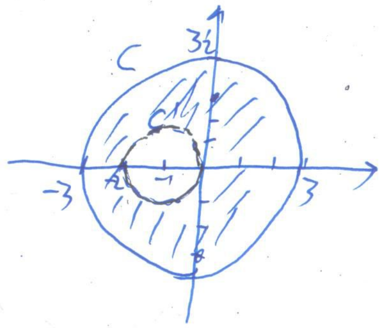
MATH 2230 Tutorial 7

1. Compute $\int_C \frac{z^2}{z+1}\,dz$ where $C = \{3e^{i\theta} \mid \theta \in [0, 2\pi)\}$

Ans: Let $C' = \{-1 + e^{i\theta} \mid \theta \in [0, 2\pi)\}$.



$\because \frac{z^2}{z+1}$ is analytic between C and C'

$\therefore \int_C \frac{z^2}{z+1}\,dz = \int_{C'} \frac{z^2}{z+1}\,dz$

$\therefore \int_{C'} \frac{z^2}{z+1}\,dz$

$= \int_0^{2\pi} \frac{1 - 2e^{i\theta} + e^{i2\theta}}{e^{i\theta}}\, i e^{i\theta}\, d\theta$

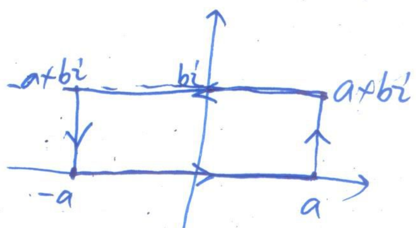
$= i \int_0^{2\pi} (1 - 2e^{i\theta} + e^{i2\theta})\, d\theta$

$= i\left(\theta - \frac{2}{i} e^{i\theta} + \frac{1}{2i} e^{2i\theta}\right)\Big|_0^{2\pi}$

$= 2\pi i$

2. (a) Let C be the path in the graph,

By integrating e^{-z^2} on C, show



$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy$$

(b) Given $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Show $\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$ $(b>0)$

Ans: (a) $\int_C e^{-z^2}dz$

$$= \int_{-a}^a e^{-x^2}dx + \int_0^b e^{-(a+iy)^2}i\,dy - \int_{-a}^a e^{-(x+bi)^2}dx - \int_0^b e^{-(-a+iy)^2}i\,dy$$

$$= 2\int_0^a e^{-x^2}dx + \int_0^b e^{-(a^2-y^2+2ayi)}i\,dy - \int_{-a}^a e^{-(x^2-b^2+2xbi)}dx - \int_0^b e^{-(a^2-2ayi-y^2)}i\,dy$$

$$= 2\int_0^a e^{-x^2}dx + e^{-a^2}\int_0^b e^{y^2}(\cos 2ay - i\sin 2ay)i\,dy$$
$$- e^{b^2}\int_{-a}^a e^{-x^2}(\cos 2xb - i\sin 2xb)dx$$
$$- e^{-a^2}\int_0^b e^{y^2}(\cos 2ay + i\sin 2ay)i\,dy$$

$$= 2\int_0^a e^{-x^2}dx + e^{-a^2}\int_0^b e^{y}(\sin 2ay + i\cos 2ay)dy$$
$$- e^{b^2}\int_{-a}^a e^{-x^2}(\cos 2xb - i\sin 2xb)dx$$
$$- e^{-a^2}\int_0^b e^{y^2}(-\sin 2ay + i\cos 2ay)dy$$

$\because e^{-z^2}$ is analytic

$\therefore$ By Cauchy-Goursat Thm

$\int_C e^{-z^2}dz = 0$

$\therefore \mathrm{Re}\left(\int_C e^{-z^2}dz\right) = 0$

$$\therefore 2\int_0^a e^{-x^2}dx + e^{-a^2}\int_0^b e^{y^2}\sin 2ay\,dy$$

$$-e^{b^2}\int_{-a}^{a} e^{-x^2}\cos 2xb\,dx$$

$$+e^{-a^2}\int_0^b e^{y^2}\sin 2ay = 0$$

$$\therefore e^{b^2}\int_{-a}^{a} e^{-x^2}\cos 2xb\,dx = 2\left(\int_0^a e^{-x^2}dx + e^{-a^2}\int_0^b e^{y^2}\sin 2ay\,dy\right)$$

$$= 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

$$\therefore \int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy$$

(b) $\because |e^{y^2}\sin 2ay| \le e^{y^2}$

$$\therefore \left|e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy\right|$$

$$\le e^{-(a^2+b^2)}\int_0^b e^{y^2}dy$$

$$\to 0 \quad \text{as } a\to\infty$$

$\therefore$ Let $a\to\infty$

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^\infty e^{-x^2}dx + 0$$

$$= e^{-b^2}\frac{\sqrt{\pi}}{2}$$

3. let C denote the positively oriented circle $|z-z_0|=R$

show $\displaystyle\int_{C_0}(z-z_0)^n dz = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{if } n\in\mathbb{Z}\setminus\{-1\} \end{cases}$

Ans: 1°. when $n=-1$

$$z=z_0+Re^{i\theta}:\ \theta\in[0,2\pi)$$

$$\int_{C_0}(z-z_0)^n dz = \int_0^{2\pi}\frac{1}{Re^{i\theta}}Re^{i\theta}i\,d\theta$$

$$=\int_0^{2\pi} i\,d\theta = 2\pi i$$

2°. when $n\neq -1$

~~$\int_{C_0}(z-z_0)^n dz$~~

$\because$ $\frac{1}{n+1}(z-z_0)^{n+1}$ is an anti-derivative of $(z-z_0)^n$ on $\mathbb{C}\setminus\{z_0\}$

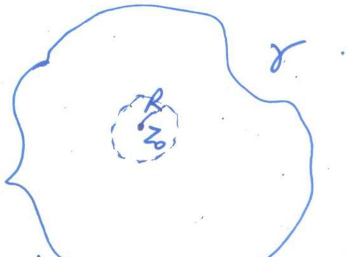
$\therefore \displaystyle\int_{C_0}(z-z_0)^n dz = 0$

4. Suppose $f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + a_n(z-z_0)^n$ for some $m, n \in N^*$

$z_0 \in Z$

Let γ be any simple closed curve on C and z_0 is inside γ

then $\int_\gamma f(z) = 2\pi i \, a_{-1}$



Ans: $\because z_0$ is inside γ

$\therefore \exists R > 0$ such that

$z_0 + Re^{i\theta}$ is inside γ for $\forall\, \theta \in [0, 2\pi)$

$\therefore$ let $\bar{\gamma} = \{z_0 + Re^{i\theta} : \theta \in [0, 2\pi)\}$

By Cauchy-Goursat Thm

$\int_\gamma f(z)dz = \int_{\bar{\gamma}} f(z)dz$ since $f(z)$ is analytic between γ and $\bar{\gamma}$

$\therefore \int_{\bar{\gamma}} f(z)dz$

$= a_{-m}\int_{\bar{\gamma}} \frac{1}{(z-z_0)^m}dz + a_{-m+1}\int_{\bar{\gamma}} \frac{1}{(z-z_0)^{m-1}}dz$

$+ \dots + a_{-1}\int_{\bar{\gamma}} \frac{1}{z-z_0}dz + a_0 \int_{\bar{\gamma}} 1\,dz$

$+ a_1 \int_{\bar{\gamma}} (z-z_0)dz + \dots + a_n \int_{\bar{\gamma}} (z-z_0)^n dz$

$= 0 + 0 + \dots + 0 + a_{-1} \cdot 2\pi i + 0 + 0 + \dots + 0$

$= a_{-1} 2\pi i$ by Q3

$\therefore \int_\gamma f(z)dz = 2\pi i \, a_{-1}$

and a_{-1} is called the residue of f at z_0

write $a_{-1} = \mathrm{Res}(f, z_0)$