MATH 2230 Tutorial 6

1. Use $e^{(1+ni)x} = e^x e^{inx} = e^x \cos nx + i e^x \sin nx$

to compute $\int_0^\pi e^x \cos nx \, dx$ and $\int_0^\pi e^x \sin nx \, dx$

Ans: $\because$ $e^{(1+ni)x} = e^x \cos nx + i e^x \sin nx$

$\therefore$ $\int_0^\pi e^{(1+ni)x} dx = \int_0^\pi e^x \cos nx \, dx + i \int_0^\pi e^x \sin nx \, dx$

$$\because \int_0^\pi e^{(1+ni)x} dx = \frac{1}{1+ni} e^{(1+ni)x} \Big|_0^\pi$$

$$= \frac{1}{1+ni}\left(e^\pi e^{in\pi} - 1\right)$$

$$= \frac{1}{1+ni}\left((-1)^n e^\pi - 1\right)$$

$$= \frac{1-ni}{1+n^2}\left[(-1)^n e^\pi - 1\right]$$

$$= \frac{(-1)^n e^\pi - 1}{1+n^2} - i \cdot \frac{n[(-1)^n e^\pi - 1]}{1+n^2}$$

$\therefore$ By equating the real part and imaginary part, we get

$$\int_0^\pi e^x \cos nx \, dx = \frac{(-1)^n e^\pi - 1}{1+n^2}$$

$$\int_0^\pi e^x \sin(nx) \, dx = -\frac{n}{1+n^2}\left[(-1)^n e^\pi - 1\right]$$

2. Compute $\int_C \frac{1}{z} dz$ where $C = \{ e^{i\theta} \mid \theta \in [0, 2\pi) \}$

Ans: write $z = e^{i\theta}$: $\theta \in [0, 2\pi)$, then $dz = ie^{i\theta} d\theta$

then by definition

$$\int_C \frac{1}{z} dz = \int_0^{2\pi} e^{-i\theta} i e^{i\theta} d\theta = \int_0^{2\pi} i \, d\theta = 2\pi i$$

• Remark: locally, we know $\frac{1}{z} = (\log z)'$

then $\int \frac{1}{z} dz = \int d(\log z)$

Even though $\frac{1}{z}$ has a ~~primitive~~ primitive function Locally,

the integration of $\frac{1}{z}$ over a closed curve may not be zero.

3. $f(z)$ is a function define on whole $\mathbb{C}$

Suppose $\exists$ an analytic function $F(z)$ such that $F'(z) = f(z)$

show $\forall$ simple closed curve γ, $\int_\gamma f(z) dz = 0$

Pf: Write $F(z) = u + iv$

$\because F(z)$ is analytic $\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$f = F' \Rightarrow f = u_x + i v_x$

$$\therefore \int_\gamma f(z) dz = \int_a^b (u_x + i v_x)(x'(t) + i y'(t)) dt$$

$$= \int_a^b [u_x x' - v_x y'] + i [v_x x' + u_x y'] dt$$

$$= \int_a^b [u_x x' + u_y y'] dt + i \int_a^b [v_x x' + v_y y'] dt$$

$$= \int_a^b \frac{d}{dt} u + i \int_a^b \frac{d}{dt} v$$

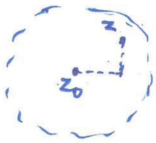
$$= u(\gamma(b)) - u(\gamma(a)) + i [v(\gamma(b)) - v(\gamma(a))]$$

$$= 0 + 0 = 0 \quad \text{since } \gamma(a) = \gamma(b)$$

4. Show that every analytic function has a local primitive function.

i.e: Let f be a function which is analytic function at z_0
then $\exists$ open set $U \ni z_0$, an analytic function $F(z)$ defined on U
such that $F'(z) = f(z)$

Pf: Since f is analytic at $z_0 = x_0 + iy_0$



$\therefore \exists\, \varepsilon > 0$ such that

$f(z)$ is differentiable on $B_\varepsilon(z_0) = \{ z_0 + re^{i\theta} \mid |r| < \varepsilon,\ \theta \in [0, 2\pi] \}$

Write $f(z) = u(x,y) + iv(x,y)$

$\forall z \in B_\varepsilon(z_0)$, $z = \bar{x} + i\bar{y}$

- define $F(z) = \int_{x_0}^{\bar{x}} (u(x,y_0) + iv(x,y_0))\, dx$
$\quad + i \int_{y_0}^{\bar{y}} (u(\bar{x},y) + iv(\bar{x},y))\, dy$
$\cancel{g(\bar{x},\bar{y}) + ih(\bar{x},\bar{y}))}$

- Now, we check $F(z)$ is differentiable on $B_\varepsilon(z_0)$ using C-R eqn.

$\cancel{\therefore \partial_{\bar{x}} g(\bar{x},\bar{y}) = u(\bar{x},y_0) + iv(\bar{x},y_0))}$

$\cancel{\partial\, g(\bar{x},\bar{y})}$

$$\int_{x_0}^{\bar{x}} u(x,y_0) + iv(x,y_0)\, dx + i \int_{y_0}^{\bar{y}} u(\bar{x},y) + iv(\bar{x},y)\, dy$$

$$= \left[\int_{x_0}^{\bar{x}} u(x,y_0) dx - \int_{y_0}^{\bar{y}} v(\bar{x},y) dy \right] + i \left[\int_{x_0}^{\bar{x}} v(x,y_0) dx + \int_{y_0}^{\bar{y}} u(\bar{x},y) dy \right]$$

$$= g(\bar{x},\bar{y}) + ih(\bar{x},\bar{y})$$

$\therefore\ \partial_{\bar{x}} g(\bar{x},\bar{y}) = u(\bar{x},y_0) - \int_{y_0}^{\bar{y}} \partial_x v(\bar{x},y)\, dy = u(\bar{x},y_0) + \int_{y_0}^{\bar{y}} \partial_y u(\bar{x},y)\, dy$
$= u(\bar{x},\bar{y})$

$\partial_{\bar{y}} g(\bar{x},\bar{y}) = 0 - v(\bar{x},\bar{y})$

$\partial_{\bar{x}} h(\bar{x},\bar{y}) = v(\bar{x},y_0) + \int_{y_0}^{\bar{y}} \partial_x u(\bar{x},y)\, dy = v(\bar{x},y_0) + \int \partial_y v(\bar{x},y) dy$
$= v(\bar{x},\bar{y})$

$\partial_{\bar{y}} h(\bar{x},\bar{y}) = 0 + u(\bar{x},\bar{y})$

- To check $\begin{cases} \partial_{\bar{x}} g = \partial_{\bar{y}} h \\ \partial_{\bar{y}} g = -\partial_{\bar{x}} h \end{cases}$, we need to use f is differentiable at $(\bar{x},\bar{y})$
$\begin{cases} \partial_{\bar{x}} u = \partial_{\bar{y}} v \\ \partial_{\bar{y}} u = -\partial_{\bar{x}} v \end{cases}$

$\therefore F(z)$ is differentiable and $F'(z) = \partial_x g + i\,\partial_x h = u + iv = f(z)$