MATH 2230 Tutorial 4

1. Find all roots of the equations

i) $\sinh z = i$

ii) $\cosh z = \frac{1}{2}$

Ans: i) $\sinh z = \frac{e^z - e^{-z}}{2} = i$.

$\therefore e^z - e^{-z} = 2i$

$\therefore e^{2z} - 1 = 2ie^z$ since $e^z \neq 0, \forall z \in \mathbb{C}$

$\therefore e^{2z} - 2ie^z - 1 = 0$

$e^z = \frac{2i \pm \sqrt{-4+4}}{2}$

$= i = e^{i(\frac{\pi}{2} + 2n\pi)}$

$\therefore z = (\frac{\pi}{2} + 2n\pi)i$

ii) $\cosh z = \frac{1}{2}$

$\frac{e^z + e^{-z}}{2} = \frac{1}{2}$

$e^{2z} - e^z + 1 = 0$

$\therefore e^z = \frac{1 \pm (-3)^{\frac{1}{2}}}{2} = \frac{1 \pm \sqrt{3}i}{2} = e^{i(\pm\frac{\pi}{3} + 2n\pi)}$

$\therefore z = (2n\pi \pm \frac{\pi}{3})i$

2. Compute:

i) $Lg(1-i)$

ii) $Lg\, i$

iii) show $Lg(i^3) \neq 3\, Lg\, i$

iv) Find all z such that $Lg\, z = \frac{\pi}{2} i$

Ans: i) $Log(1-i) = Log\left(\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)\right) = Log\left(\sqrt{2}\, e^{i(-\frac{\pi}{4})}\right)$

$= lg\sqrt{2} + i\left(-\frac{\pi}{4}\right) = \frac{1}{2} lg\, 2 - \frac{\pi}{4} i$

ii) $lg\, i = lg\, e^{i(\frac{\pi}{2} + 2n\pi)} = i\left(\frac{\pi}{2} + 2n\pi\right)$

iii) $log(i^3) = log(-i) = Log\left(e^{i(-\frac{\pi}{2})}\right) = -\frac{\pi}{2} i$

$3 Log\, i = 3 Log\left(e^{i(\frac{\pi}{2})}\right) = 3\left(\frac{\pi}{2} i\right) = \frac{3}{2}\pi i$

$\therefore Log(i^3) \neq 3 Log\, i$

iv) $lg\, z = lg|z| + i(\arg z + 2n\pi)$

$\because lg\, z = \frac{\pi}{2} i$

$\therefore \begin{cases} lg|z| = 0 \\ i(\arg z + 2n\pi) = \frac{\pi}{2} i \end{cases} \Rightarrow \begin{cases} |z| = 1 \\ \arg z = \frac{\pi}{2} - 2n\pi \end{cases}$

$\therefore z = |z| e^{i(\arg z)} = e^{i(\frac{\pi}{2} - 2n\pi)} = e^{i\frac{\pi}{2}} = i$

3. Let $z \neq 0$, let $\Theta \in (-\pi, \pi]$ such that $|z| e^{i\Theta} = z$

Let n be a positive integer.

- show $\operatorname{Log}(z^{\frac{1}{n}}) = \frac{1}{n} \lg r + i \frac{\Theta + 2(pn+k)\pi}{n}$ where $p, k \in \mathbb{Z}$

  and $\frac{1}{n} \log z = \frac{1}{n} \lg r + i \frac{\Theta + 2q\pi}{n}$ where $q \in \mathbb{Z}$

- Then show $\operatorname{Log}(z^{\frac{1}{n}}) = \frac{1}{n} \operatorname{Log} z$

Ans: $\operatorname{Log}(z^{\frac{1}{n}}) = \operatorname{Log}(r^{\frac{1}{n}} e^{i(\frac{\Theta + 2k\pi}{n})})$

$$= \frac{1}{n} \log r + i \left(\frac{\Theta + 2k\pi}{n} + 2p\pi \right)$$

$$= \frac{1}{n} \log r + i \frac{\Theta + 2(pn+k)\pi}{n}$$

$$\frac{1}{n} \log z = \frac{1}{n} (\lg r + i(\Theta + 2q\pi))$$

$$= \frac{1}{n} \log r + i \frac{\Theta + 2q\pi}{n}$$

- In order to show $\operatorname{Log}(z^{\frac{1}{n}}) = \frac{1}{n} \operatorname{Log} z$

  we need to show

  $$\{pn+k \mid p, k \in \mathbb{Z}\} = \{q \mid q \in \mathbb{Z}\}$$

  since n is a fixed integer

  $\therefore \forall q \in \mathbb{Z}, \exists p, k \in \mathbb{Z}$ such that $q = pn + k$

  $\therefore \{q \mid q \in \mathbb{Z}\} \subseteq \{pn+k \mid p, k \in \mathbb{Z}\}$

  $\because pn+k$ is an integer

  $\therefore \{pn+k \mid p, k \in \mathbb{Z}\} \subseteq \{q \mid q \in \mathbb{Z}\}$

  $\therefore \{pn+k \mid p, k \in \mathbb{Z}\} = \{q \mid q \in \mathbb{Z}\}$

4. Compute:

i) $(1+i)^i$

ii) $(-i)^i$

iii) i^{-2i}

Ans: i) $(1+i)^i = e^{i\lg(1+i)} = e^{i\lg[\sqrt{2}(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)]} = e^{i\lg[\sqrt{2}e^{i\frac{\pi}{4}}]}$

$= e^{i[\lg\sqrt{2}+i(\frac{\pi}{4}+2n\pi)]} = e^{2n\pi-\frac{\pi}{4}} e^{i\frac{1}{2}\lg 2}$

ii) $(-i)^i = e^{i\lg(-i)} = e^{i\lg(e^{i(-\frac{\pi}{2})})}$

$= e^{i[0+i(-\frac{\pi}{2}+2n\pi)]} = e^{\frac{\pi}{2}+2n\pi}$

iii) $i^{-2i} = e^{-2i\lg i} = e^{-2i\lg(e^{i\frac{\pi}{2}})} = e^{-2i[0+i(\frac{\pi}{2}+2n\pi)]}$

$= e^{\pi+4n\pi}$

5. let $c = a+bi$ be a complex number

What additional restriction must be placed on the constant c so that $|i^c|$ are all the same?

Ans: $i^c = e^{c\lg i} = e^{c[0+i(\frac{\pi}{2}+2n\pi)]} = e^{(a+bi)[i(\frac{\pi}{2}+2n\pi)]}$

$= e^{-(\frac{\pi}{2}+2n\pi)b + ia(\frac{\pi}{2}+2n\pi)}$

$\therefore |i^c| = e^{-(\frac{\pi}{2}+2n\pi)b}$

$\therefore |i^c|$ are all the same

$\Leftrightarrow b=0$

$\Leftrightarrow c$ is real.