MATH 2230 A

Tutorial 1: Sept 13, 2017

1. Show i) $1+z+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}$

ii) $1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \dfrac{1}{2}+\dfrac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)}$

Pf: i) $(1+z+\cdots+z^n)(1-z) = 1+z+\cdots+z^n - z - \cdots - z^{n+1} = 1-z^{n+1}$

$\therefore\ 1+z+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}$ when $z\neq 1$

ii) Let $z = e^{i\theta}$

$$\begin{aligned}
\therefore\ & 1+\cos\theta+\cdots+\cos n\theta \\
&= Re[1+z+\cdots+z^n] \\
&= Re\left[\frac{1-z^{n+1}}{1-z}\right] = Re\left[\frac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{1-\cos\theta - i\sin\theta}\right] \\
&= Re\left[\frac{(1-\cos(n+1)\theta - i\sin(n+1)\theta)(1-\cos\theta+i\sin\theta)}{(1-\cos\theta-i\sin\theta)(1-\cos\theta+i\sin\theta)}\right] \\
&= \frac{[1-\cos(n+1)\theta](1-\cos\theta)+\sin(n+1)\theta\cdot\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} \\
&= \frac{1-\cos(n+1)\theta-\cos\theta+\cos\theta\cos(n+1)\theta+\sin\theta\sin(n+1)\theta}{2-2\cos\theta} \\
&= \frac{1-\cos\theta-\cos(n+1)\theta+\cos n\theta}{2-2\cos\theta} \\
&= \frac{1}{2}+\frac{-\cos[(n+\frac{1}{2})\theta+\frac{\theta}{2}]+\cos[(n+\frac{1}{2})\theta-\frac{\theta}{2}]}{4\sin^2\frac{\theta}{2}} \quad \text{where } 1-\cos\theta = 2\sin^2\frac{\theta}{2} \\
&= \frac{1}{2}+\frac{2\sin(n+\frac{1}{2})\theta\,\sin\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} \\
&= \frac{1}{2}+\frac{\sin[(n+\frac{1}{2})\theta]}{2\sin\frac{\theta}{2}}
\end{aligned}$$

2. Polar form: $z = re^{i\theta}$

∴ $(e^{i\theta})^n = e^{i(n\theta)}$

∴ $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

When $n=3$, $(\cos\theta+i\sin\theta)^3 = (\cos\theta+i\sin\theta)(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta)$

$= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta + i(\cos^2\theta\sin\theta - \sin^3\theta + 2\sin\theta\cos^2\theta)$

$= \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

∴ $\begin{cases} \cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta \\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta \end{cases}$

3. Roots: solve i) $(-8-8\sqrt{3}i)^{\frac{1}{4}}$

ii) $(1-i)^{\frac{1}{2}}$

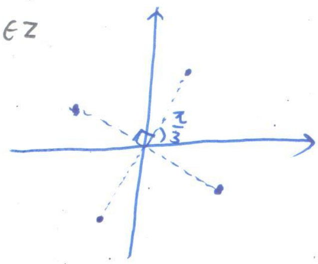
iii) $(-i)^{\frac{1}{3}}$

Ans: i) $(-8-8\sqrt{3}i)^{\frac{1}{4}} = \left[16\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right]^{\frac{1}{4}} = \left[16e^{i(\frac{4}{3}\pi+2n\pi)}\right]^{\frac{1}{4}}$, $n\in\mathbb{Z}$

$= 2e^{i(\frac{\pi}{3}+\frac{n}{2}\pi)}$, $n\in\mathbb{Z}$

$= 2e^{i\frac{\pi}{3}}, 2e^{i\frac{5}{6}\pi}, 2e^{i\frac{4}{3}\pi}$ or $2e^{i\frac{11}{6}\pi}$

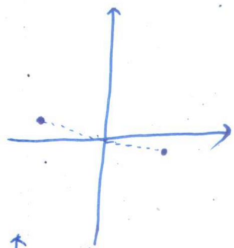
$= 1+\sqrt{3}i, -\sqrt{3}+i, -1-\sqrt{3}i$ or $\sqrt{3}-i$



ii) $(1-i)^{\frac{1}{2}} = \left[\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\right]^{\frac{1}{2}} = \left[2^{\frac{1}{2}}e^{i(-\frac{\pi}{4}+2n\pi)}\right]^{\frac{1}{2}}$

$= 2^{\frac{1}{4}}e^{i(-\frac{\pi}{8}+n\pi)}$

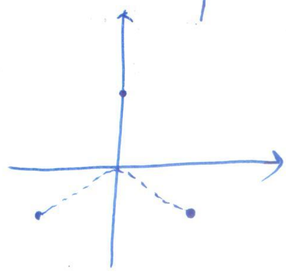
$= 2^{\frac{1}{4}}e^{i(-\frac{3}{8})}$ or $2^{\frac{1}{4}}e^{i\frac{7}{8}\pi}$



iii) $(-i)^{\frac{1}{3}} = (0-i\cdot 1)^{\frac{1}{3}} = \left(e^{i(\frac{3}{2}\pi+2n\pi)}\right)^{\frac{1}{3}}$

$= e^{i(\frac{\pi}{2}+\frac{2n}{3}\pi)}$

$= e^{i\frac{\pi}{2}}, e^{i\frac{7}{6}\pi}$ or $e^{i\frac{11}{6}\pi}$.



4. a) For $az^2+bz+c=0$ $(a\neq 0)$
derive the quadratic formula

$$z=\frac{-b\pm(b^2-4ac)^{\frac{1}{2}}}{2a}$$

b) Find the roots of $z^2+2z+(1-i)=0$

Ans: (a) $az^2+bz+c=0$ $(a\neq 0)$

$$a(z^2+\frac{b}{a}z)=-c$$

$$z^2+\frac{b}{a}z=-\frac{c}{a}$$

$$z^2+\frac{b}{a}z+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

$$(z+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$$

$$\therefore z+\frac{b}{2a}=\pm\frac{(b^2-4ac)^{\frac{1}{2}}}{2a}$$

$$\therefore z=\frac{-b\pm(b^2-4ac)^{\frac{1}{2}}}{2a}$$

(b) $z^2+2z+(1-i)=0$

$$z=\frac{-2\pm(4-4(1-i))^{\frac{1}{2}}}{2}$$

$$=\frac{-2\pm(4i)^{\frac{1}{2}}}{2}$$

$$=-1\pm\frac{[4e^{i(\frac{\pi}{2}+2n\pi)}]^{\frac{1}{2}}}{2}$$

$$=-1\pm e^{i(\frac{\pi}{4}+n\pi)}$$

$$=-1\pm(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)$$

$$=(-1+\frac{\sqrt{2}}{2})+\frac{\sqrt{2}}{2}i \quad \text{or} \quad (-1-\frac{\sqrt{2}}{2})-\frac{\sqrt{2}}{2}i$$

5. Regions in the complex plane:

(a) $|z-1+i| \leq 1$

(b) $|2z+3| \geq 4$

(c) $Im(z) = 1$

(d) $Re(z) \geq 2$

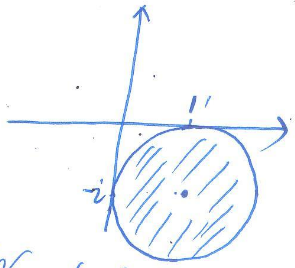
(e) $0 \leq \arg z \leq \frac{\pi}{4}$

Ans: (a) Let $z = 1-i+re^{i\theta}$.

$\because |z-1+i| \leq 1$

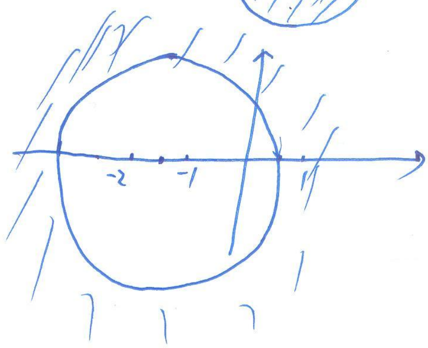
$\Rightarrow |1-i+re^{i\theta}-1+i| \leq 1$

$\Rightarrow r \leq 1$



(b) $|2z+3| \geq 4$

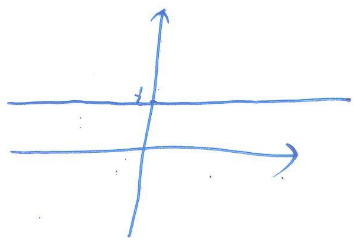
$|z-(-\frac{3}{2})| \geq 2$



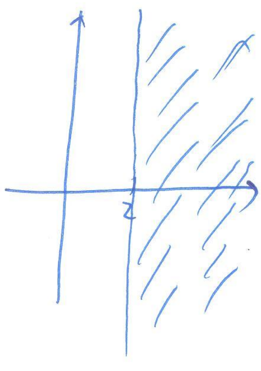
(c) $Im(z) = 1$

Let $z = x+iy$

$\because Im(z) = y = 1$



(d) $Re(z) \geq 2$



(e) $0 \leq \arg z \leq \frac{\pi}{4}$