Pf (of Rouché's Thm)

We may assume C is positively oriented.

Then $|f(z)| > |g(z)| \geq 0$

and $|f(z)+g(z)| \geq |f(z)| - |g(z)| > 0$ $\Big\}$ for $z \in C$

$\therefore$ $Z_f = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$ and (by argument principle)

$$Z_{f+g} = \frac{1}{2\pi i} \int_C \frac{(f+g)'}{(f+g)}(z)\, dz$$

$$= \frac{1}{2\pi i} \int_C \left(\frac{f'(z)}{f(z)} + \frac{F'(z)}{F(z)} \right) dz$$

where $F(z) = 1 + \frac{g(z)}{f(z)}$

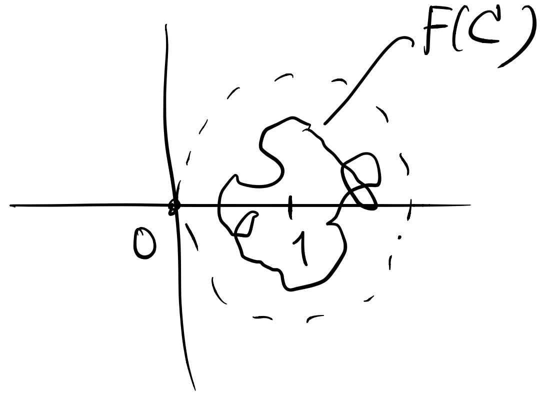
meromorphic inside C and analytic & $\neq 0$ on C.

$$\Rightarrow Z_{f+g} - Z_f = \frac{1}{2\pi i} \int_C \frac{F'(z)}{F(z)} dz$$

$$= \frac{1}{2\pi} \Delta_C \arg F(z).$$

Note that $|F(z)-1| = \left|\frac{g(z)}{f(z)}\right| < 1$

$\therefore$ $F(C)$ cannot surround the origin $w=0$



$\therefore$ $\frac{1}{2\pi}\Delta_C \arg F(z) = 0$

Hence $\quad Z_{f+g} = Z_f$. $\quad$ ※

eg: (Application : determine "number of roots")

Consider eqt. $\quad z^4 + 3z^3 + 6 = 0$ inside $|z| = 2$

Let $f(z) = 3z^3$ and $g(z) = z^4 + 6$

Then on $C = \{|z| = 2\}$,

$$|f(z)| = |3z^3| = 3 \cdot 2^3 = 24$$

and $\quad |g(z)| = |z^4 + 6| \leq |z|^4 + 6 = 2^4 + 6 = 22$

$$< 24 = |f(z)| \quad (\forall z \in C.)$$

Hence Rouché's Thm $\Rightarrow$

$f(z)+g(z)$ & $f(z)$ have the same # of zeros inside $\{|z|=2\}$.

Since $f(z)=3z^3$ has 3 zeros (counting multiplicity)

we see that

$$z^4+3z^3+6 = f(z)+g(z) \text{ has } 3$$

zeros inside $\{|z|=2\}$.

i.e. 3 roots inside $\{|z|=2\}$.

(counting multiplicities).

Remark: Rouché's Thm can be used to prove the fundamental theorem of algebra:

For $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \quad (a_n \neq 0)$

Let $f(z) = a_n z^n$

$$g(z) = a_{n-1} z^{n-1} + \cdots + a_0$$

Then for $R > \max\left\{1, \dfrac{|a_{n-1}|+\cdots+|a_0|}{|a_n|}\right\}$, we have $\forall\ |z| = R$,

$$|f(z)| = |a_n| R^n, \text{ and}$$

$$\begin{aligned} |g(z)| &\leq |a_{n-1}| |z|^{n-1} + \cdots + |a_0| \\ &= |a_{n-1}| R^{n-1} + \cdots + |a_0| \\ &\leq R^{n-1} (|a_{n-1}| + \cdots + |a_0|) \quad (\text{by } R > 1) \\ &< |a_n| R^n = |f(z)| \end{aligned}$$

Hence Rouché's Thm $\Rightarrow$

$P(z) = f(z) + g(z)$ has the same number of zeros (counting multiplicities) as $f(z) = a_n z^n$.

$\Rightarrow$ $P(z)$ has n roots (counting multiplicities).