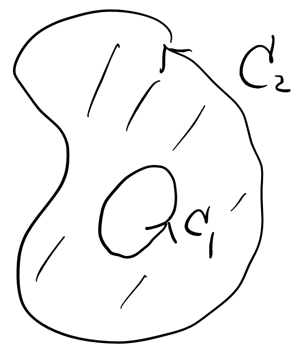
Cor (Principle of deformation of paths)

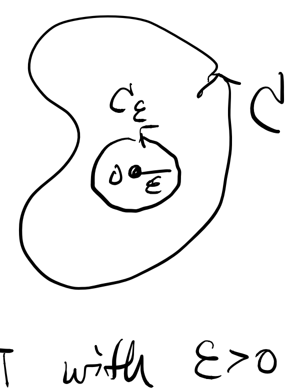
Let C_1 & C_2 be positively oriented simple closed contours, where C_1 is interior to C_2. If f is analytic in the closed region consisting of C_1 & C_2 and all points between them, then



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz .$$

Pf: By Thm: $\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$ #

eg: Let C = any positively oriented simple closed contour surrounding the origin



Then $\int_C \frac{dz}{z} = 2\pi i$

Pf: Choose $C_\varepsilon : z = \varepsilon e^{i\theta}$, $0 \le \theta \le 2\pi$ with $\varepsilon > 0$ small enough s.t. $B_\varepsilon(0)$ is interior to C.

Then by the principle of defumation of paths,

$$\int_C \frac{dz}{z} = \int_{C_\varepsilon} \frac{dz}{z}$$ since $f(z) = \frac{1}{z}$ is analytic between and on C & C_ε.

$$= \int_0^{2\pi} \frac{d(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}$$

$$= \int_0^{2\pi} i\,d\theta = 2\pi i$$

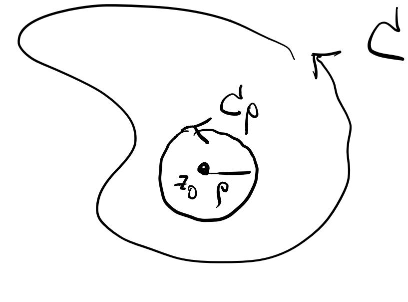
\#

§54 Cauchy Integral Fumula

Thm: Let f be analytic everywhere inside and on a simple closed contour C in positive orientation. If z_0 is any point interior to C, then

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz}$$ Cauchy Integral Fumula

Pf: Since z_0 is interior to C, $\forall \rho > 0$ small enough, $B_\rho(z_0)$ is interior to C.



Let $C_\rho = \partial B_\rho(z_0)$ parametried by $\begin{cases} z = z_0 + \rho e^{i\theta} \\ 0 \le \theta \le 2\pi \end{cases}$

Then by principle of deformation of paths, we have

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$$

$$= \int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})}{\rho e^{i\theta}}\, d(z_0+\rho e^{i\theta})$$

$$= 2\pi i \left[\frac{1}{2\pi} \int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta \right]$$

$$\longrightarrow 2\pi i\, f(z_0) \quad \text{as } \rho \to 0$$

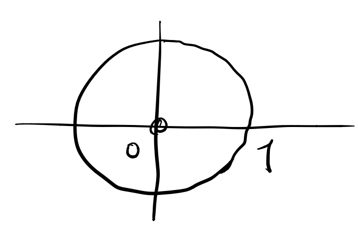
(since f is cts.)

$$\therefore \quad f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\,dz.$$

#

eg: Let $f(z) = \dfrac{\cos z}{z^2+9}$, $C: z = e^{i\theta}$, $0 \le \theta \le 2\pi$ (positive oriented unit circle)

Since $z = \pm 3i$ are not interior to C, then $f(z)$



is analytic interior in and on C.

$$\frac{1}{9} = f(0) \overset{\text{(Cauchy integral formula)}}{=} \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,dz = \frac{1}{2\pi i}\int_C \frac{\cos z}{z(z^2+9)}\,dz$$

or $$\int_C \frac{\cos z}{z(z^2+9)}\,dz = \frac{2\pi i}{9}.$$

§55 An Extension of the Cauchy Integral Formula

Notation: $f^{(n)}(z_0)$ denotes the n-th derivative of f at z_0, where $f^{(0)}(z_0) = f(z_0)$.

$$\left(f^{(n)}(z_0) = \frac{d}{dz} f^{(n-1)} \Big|_{z=z_0} \right)$$

Thm: Let f be analytic inside and on a simple closed contour C in positive orientation. If z_0 is any point interior to C, then $\forall\, n = 0, 1, 2, \cdots$

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}}$$

Cauchy Integral Formula.

Application:

eg1: If C = positive oriented unit circle.

Then $\int_C \frac{\exp(2z)}{z^4} dz = \int_C \frac{\exp(2z)}{(z-0)^{3+1}} dz$

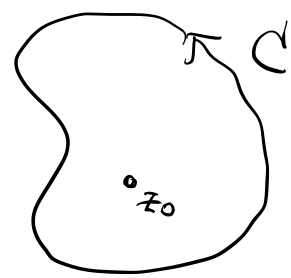
$$= \frac{2\pi i}{3!} f^{(3)}(0)$$

where $f(z) = \exp(2z)$
$= e^{2z}$.

$$f^{(3)}(z) = 8e^{2z}$$

$$\therefore \int_C \frac{\exp(2z)}{z^4} dz = \frac{2\pi i}{3!} \cdot 8 = \frac{8\pi i}{3}$$

eg2: C = positively oriented simple closed contour, z_0 interior to C.



Then applying the Cauchy Integral Formula to $f(z) \equiv 1$.

For $n=0$, $\quad 1 = \frac{1}{2\pi i} \int_C \frac{1}{z-z_0} dz$

For $n \geq 1$ $\quad 0 = \frac{n!}{2\pi i} \int_C \frac{1}{(z-z_0)^{n+1}} dz$

i.e. $\displaystyle\int_C \frac{1}{(z-z_0)^{n+1}}\,dz = \begin{cases} 2\pi i & \text{for } n=0 \\ 0 & \text{if } n\geq 1. \end{cases}$

Note : Replace the dummy index of the integral in the Cauchy Integral Formula by s and then let z_0 be a general point z interior to C. Then

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+1}}, \quad n=0,1,2,\dots}$$

§57 Some Consequences of the (Extension of) Cauchy Integral Formula.

Thm 1 : If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.

(Pf : Clearly follows from Cauchy Integral Formula.)

Cor: If $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z = x+iy$, then u and v have continuous partial derivatives of all order at that point.

Thm2 (Morera Theorem)

Let f be continuous on a domain D. If

$\int_C f(z)dz = 0$ for every closed contour C in D,

then f is analytic throughout D.

Pf: If $\int_C f(z)dz = 0$, $\forall$ closed contour C,

then f has an anti-derivative F in D,

ie $F'(z) = f(z)$, $\forall z \in D$

$\Rightarrow$ F is analytic in D.

By Thm 1, $f = F'$ is analytic in D. #