Thm Let C be a contour of length L, and $f(z)$ be a piecewise continuous function on C.

Suppose $M>0$ is constant such that

$$|f(z)| \leq M, \quad \forall z \in C.$$

Then

$$\left| \int_C f(z)\,dz \right| \leq ML.$$

Pf: Parametrize C by $z=z(t)$, $a\leq t\leq b$.

Then $\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$

Lemma $\Rightarrow$

$$\left| \int_C f(z)dz \right| = \left| \int_a^b f(z(t)) z'(t)\,dt \right|$$

$$\leq \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

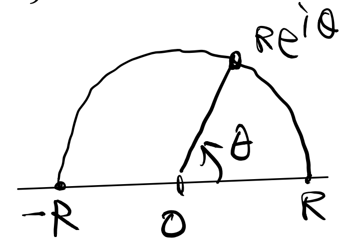
$$\leq M \int_a^b |z'(t)|\,dt$$

$$= ML \quad ※$$

(Note: It is convenient to write $\left|\int_C f(z)dz\right| \leq \int_C |f(z)|\,|dz|$ in the understanding that $|dz| = |z'(t)|\,dt$.)

eg Let C_R = semicircle: $z = Re^{i\theta}$, $0 \le \theta \le \pi$

for $R > 3$.



Show that

$$\lim_{R \to +\infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)}\, dz = 0.$$

Pf: For $R>3$, we have on C_R that

$$\begin{cases} |z+1| \le |z| + 1 = R+1 \\ |z^2+4| \ge |z|^2 - 4 = R^2 - 4 \ (>0) \\ |z^2+9| \ge |z|^2 - 9 = R^2 - 9 \ (>0) \end{cases}$$

$$\Rightarrow \left| \frac{z+1}{(z^2+4)(z^2+9)} \right| \le \frac{R+1}{(R^2-4)(R^2-9)}, \quad \forall z \in C_R$$

Hence the thm $\Rightarrow$

$$\left| \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz \right| \le \frac{R+1}{(R^2-4)(R^2-9)} \cdot \text{length of } C_R$$

$$= \frac{\pi R(R+1)}{(R^2-4)(R^2-9)} \to 0 \text{ as } R \to +\infty$$

#

§48 Antiderivatives

Def: Let $f(z)$ be a cpx-valued cts. function in a domain D. Then the antiderivative of $f(z)$ on D is a function $F(z)$ (defined on D) s.t.

$$F'(z) = f(z), \quad \forall z \in D.$$

Notes: (i) An antiderivative is an analytic function.

(ii) An antiderivative of a given function is unique up to an additive constant:

i.e. if F and G are antiderivative of f then $F - G$ is a constant function

(since domain D is connected by defn.)

Thm: Suppose that a function $f(z)$ is cts in a domain D. Then the following statements are equivalent:

(a) $f(z)$ has an antiderivative $F(z)$ throughout D.

(b) $\forall$ contours C_1 & C_2 (lying entirely in D) with the same initial and end points,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

(c) $\forall$ closed contour C (lying entirely in D),

$$\int_C f(z)\,dz = 0.$$

If any of the above statements true, then the integral in (b) is given by

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1),$$

where F is the antiderivative given in (a).

In this case, we denote

$$\boxed{\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)}$$

eg1: Let $f(z) = e^{\pi z}$ & $F(z) = \frac{1}{\pi} e^{\pi z}$ on $\mathbb{C}$

Then $F'(z) = f(z)$, $\forall z \in \mathbb{C}$.

$\Rightarrow$ $f(z)$ has antiderivative $F(z)$ on $\mathbb{C}$

$\Rightarrow$ $\forall$ contour C with initial point z_1 and end point z_2,

$$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz$$

$$= F(z)\Big|_{z_1}^{z_2} = \frac{1}{\pi} e^{\pi z}\Big|_{z_1}^{z_2}$$

$$= \frac{1}{\pi}\left(e^{\pi z_2} - e^{\pi z_1}\right)$$

<u>eg2</u> $f(z) = \frac{1}{z^n}$ on $\mathbb{C}\setminus\{0\}$, $n = 2, 3, 4, \cdots$

Note that $F(z) = -\frac{1}{(n-1)z^{n-1}}$ on $\mathbb{C}\setminus\{0\}$

satisfies $F'(z) = \frac{1}{z^n} = f(z)$, $\forall z \in \mathbb{C}\setminus\{0\}$

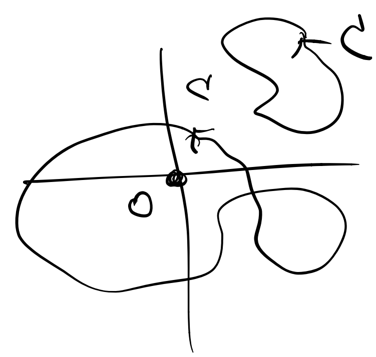
$\Rightarrow$ by part (b) of the thm, we have

$$\int_C \frac{1}{z^n}\,dz = 0, \quad \forall \text{ closed contour } C \text{ in } \mathbb{C}\setminus\{0\}$$

In particular

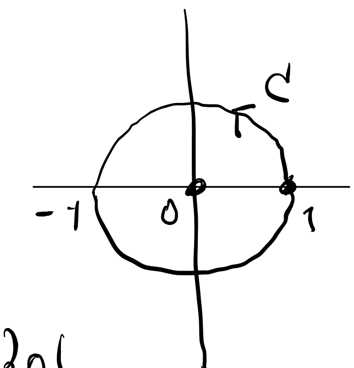
$$\int_{\text{unit circle}} \frac{1}{z^n}\,dz = 0$$

$\forall n = 2, 3, \cdots$



eg3 However, we have seen $\displaystyle\int_{\text{unit circle}} \frac{dz}{z} = 2\pi i \neq 0$.

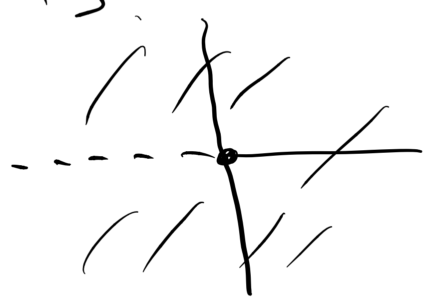
what happen?



According to the thm,

$f(z) = \frac{1}{z}$ on $\mathbb{C}\setminus\{0\}$

has no antiderivative on $\mathbb{C}\setminus\{0\}$



Note any branch of $\log z$ can only defined on $\mathbb{C}\setminus$ "ray". No "diff" $\log z$ can be defined on $\mathbb{C}\setminus\{0\}$.

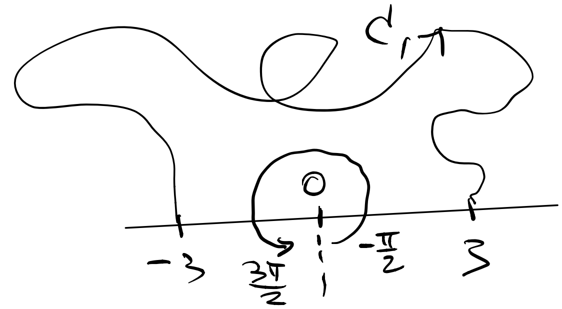
So $\log z$ (for any branch), even with

$$\frac{d}{dz}\log z = \frac{1}{z},$$

is <u>not</u> an antiderivate of $\frac{1}{z}$ on the whole $\mathbb{C}\setminus\{0\}$.

eg4: $\int_{C_1} z^{1/2}\,dz$, where C_1 = any contour from $z_1=-3$ to $z_2=3$

with $C_1 \setminus \{-3,3\} \subset \{z=x+iy : y>0\}$



and $z^{1/2}$ is the branch of $z^{1/2}$ given by

$$-\frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

Then C_1 is completely contained in the domain of the branch.

Note that in the domain of branch, we have antiderivate $F(z) = \frac{2}{3} z^{3/2}$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

Hence $\displaystyle\int_{C_1} z^{1/2}\,dz = \frac{2}{3} z^{3/2} \Big|_{-3}^{3}$

$$= \frac{2}{3} \exp\left[\frac{3}{2}\log z\right] \Big|_{-3}^{3}$$

$$= \frac{2}{3} \exp\left[\frac{3}{2}(\ln|z| + i\theta)\right]\Big|_{-3}^{3}$$

$$-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

In this branch

$$-3 = 3e^{i\pi} \quad \text{and} \quad 3 = 3e^{i\cdot 0}$$

$$\therefore \int_{C_1} z^{1/2}\,dz = \frac{2}{3}\left\{\exp\left[\frac{3}{2}(\ln 3 + i0)\right] - \exp\left[\frac{3}{2}(\ln 3 + i\pi)\right]\right\}$$

(check)

$$= 2\sqrt{3}\left(1 - e^{i\frac{3\pi}{2}}\right) = 2\sqrt{3}(1+i) \quad \#$$

§49 Proof of the Theorem

(a) $\Rightarrow$ (b)

If C is a smooth arc from z_1 to z_2 & parametized by $z = z(t)$, $a \le t \le b$. ($z(a) = z_1$, $z(b) = z_2$)

Then
$$\frac{d}{dt} F(z(t)) = F'(z(t))\,z'(t)$$
$$= f(z(t))\,z'(t)$$

$$\therefore \int_C f(z)\,dz = \int_a^b \left[\frac{d}{dt} F(z(t))\right] dt$$
$$= F(z(b)) - F(z(a))$$
$$= F(z_2) - F(z_1)$$

If C is piecewise smooth : $C = C_1 + \cdots + C_N$

with C_i smooth arc, $\forall i$, joining z_i to z_{i+1}

Then $\int_{C_i} f(z)dz = F(z_{i+1}) - F(z_i)$, $\forall i = 1, \cdots, N$

$$\Rightarrow \int_C f(z)dz = \sum_{i=1}^{N} \int_{C_i} f(z)dz$$

$$= \sum_{i=1}^{N} [F(z_{i+1}) - F(z_i)]$$

$$= F(z_{N+1}) - F(z_1)$$

(note: z_1 = initial point of C
z_{N+1} = end point of C).

(This also proved the required formula

$$\int_{z_1}^{z_2} f(z)dz = F(z)\Big|_{z_1}^{z_2}$$)

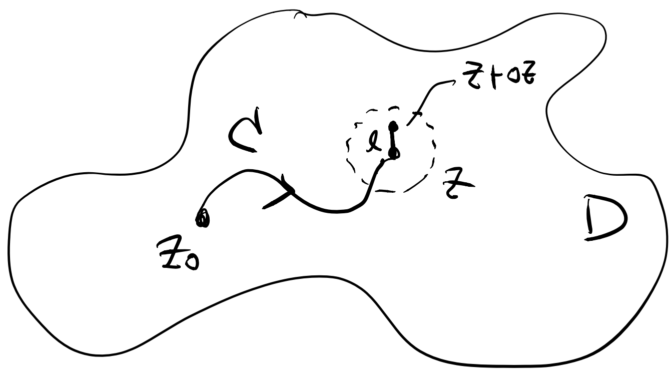
(b) ⇒ (a) Fix any $z_0 \in D$.

Then for any $z \in D$, define

$$F(z) = \int_{z_0}^{z} f(z)dz$$

which is well-defined because of the assumption (b).

For $|\Delta z|$ small, we can choose a path as in the figure



to see that

$$F(z+\Delta z)-F(z) = \int_{z_0}^{z+\Delta z} f(z)\,dz - \int_{z_0}^{z} f(z)\,dz$$

$$= \left(\int_C f(z)\,dz + \int_\ell f(z)\,dz\right) - \left(\int_C f(z)\,dz\right)$$

$$= \int_\ell f(z)\,dz$$

$$\Rightarrow \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_\ell f(s)\,ds - f(z)$$

$$= \int_\ell \left(\frac{f(s)-f(z)}{\Delta z}\right) ds$$

since $\int_\ell ds = \int_z^{z+\Delta z} ds = \Delta z$ (check)

Since f is cts, we have

$\forall \varepsilon>0, \exists \delta>0$ s.t. $|f(s)-f(z)|<\varepsilon, \forall |s-z|<\delta$

Therefore, for $|\Delta z|<\delta$, we have

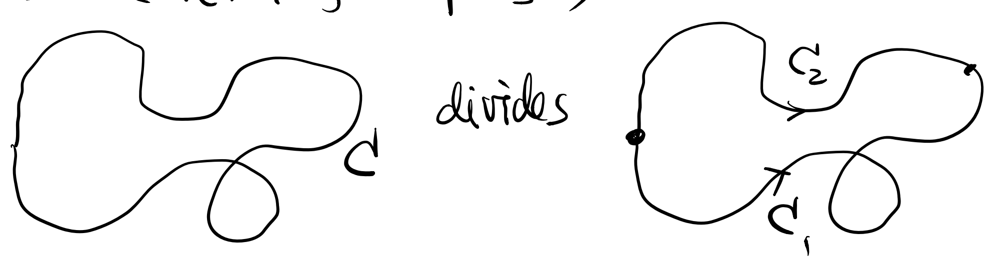
$$\left| \int_l \frac{f(s)-f(z)}{\Delta z} ds \right| \leq \frac{\varepsilon}{\Delta z} \text{ length of } l = \varepsilon$$

$\therefore$ $\forall \varepsilon>0, \exists \delta>0$ s.t.

$$\left| \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) \right| < \varepsilon, \quad \forall |\Delta z|<\delta.$$

$\therefore$ $F'(z)=f(z), \quad \forall z \in D.$

(b) $\Rightarrow$ (c) (Sketch of the proof)



$$C = C_1 - C_2$$

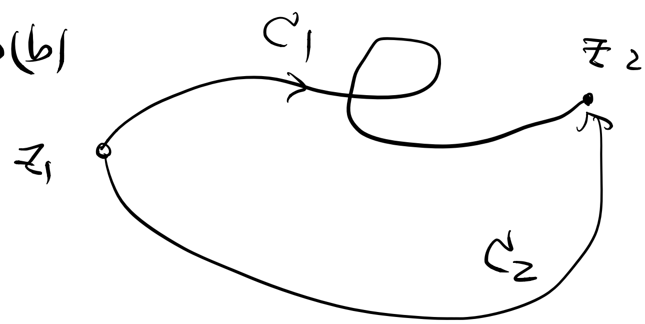
Then $\int_C f(z)dz = \int_{C_1 - C_2} f(z)dz$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)dz$$

$= 0$ since C_1, C_2 have the

same initial and end points (by (b))

(c) $\Rightarrow$ (b)



$\Rightarrow C = C_1 - C_2$ is a closed contour.

(c) $\Rightarrow$ $\displaystyle\int_C f(z)\,dz = 0$

$$\parallel$$

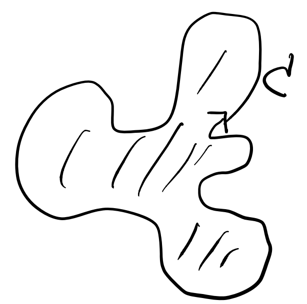
$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

#

§50 Cauchy - Goursat Theorem

Thm (Cauchy - Goursat Thm) If a function f is analytic at all point interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0.$$



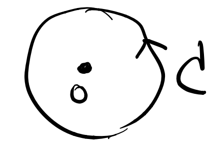
eg: C = simple closed contour in $\mathbb{C}$.

Since $\sin(z^2)$ is entire, $\sin(z^2)$ satisfies all the condition of the Cauchy - Goursat Thm. Hence

$$\int_C \sin(z^2)\,dz = 0.$$

eg: $f(z) = \frac{1}{z}$ is not analytic at $z=0$ which is interior to the unit circle $C: z = e^{i\theta},\ 0 \le \theta \le 2\pi$

and $$\int_C \frac{dz}{z} = 2\pi i \neq 0$$



Pf of the Cauchy-Goursat Thm under an additional condition that $f'(z)$ is continuous at all point interior to and on the simple closed contour C.
(This is the original Cauchy Theorem.)

Let $f(z) = u(x,y) + i\,v(x,y)$

$$C: \quad z = z(t) = x(t) + i\,y(t), \quad a \le t \le b.$$

Then $\displaystyle\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$

$$= \int_a^b (u + iv)(x' + iy')\,dt$$

$$= \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Since f' cts $\Rightarrow$ u_x, u_y, v_x, v_y are cts

Hence Green's thm $\Rightarrow$

$$\begin{cases} \displaystyle\int_C u\,dx - v\,dy = \iint_R (-u_y - v_x)\,dx\,dy \\ \displaystyle\int_C v\,dx + u\,dy = \iint_R (-v_y + u_x)\,dx\,dy \end{cases}$$

where R = region bounded by C.

Then CR-eqt $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$$\Rightarrow \int_C u\,dx - v\,dy = \int_C v\,dx + u\,dy = 0 \quad \#$$

§5.1 Proof of the Thm (without the additional condition)

(Omitted.)