Ch4 Integrals

§41 & 42 Derivatives and Definite Integrals of function $w(t)$

Def: If $w(t) = u(t) + iv(t)$ is a cpx-valued function of a real variable $t \in (a,b)$, then the derivative

$$\frac{d}{dt} w(t) = w'(t) = u'(t) + i v'(t)$$

(where u, v are real & imaginary parts of w)

Def: If $w(t) = u(t) + iv(t)$ is a cpx-valued function of a real variable $t \in [a,b]$, then the definite integral of $w(t)$ over $[a,b]$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$$

(provided the individual integrals $\int u$, $\int v$ exist.)

Thus

$$\begin{cases} \operatorname{Re} \int_a^b w(t)\,dt = \int_a^b \operatorname{Re}[w(t)]\,dt \\ \operatorname{Im} \int_a^b w(t)\,dt = \int_a^b \operatorname{Im}[w(t)]\,dt \end{cases}$$

Prop:

(1) $\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$, for $a \le c \le b$

(2) Fundamental Theorem of Calculus

If $W'(t) = w(t)$ for $t \in [a,b]$, then

$$\boxed{\int_a^b w(t)\,dt = W(b) - W(a)}$$

$$= [W(t)]_a^b = W(t)\Big|_a^b$$

$$\left(\int_a^b \left(\frac{dW}{dt}\right) dt = W(t)\Big|_a^b \quad \text{or} \quad \int_a^b dW = W(t)\Big|_a^b \right)$$

eg 2 $\quad \frac{d}{dt}\left(\frac{e^{it}}{i}\right) = e^{it}$

$$\Rightarrow \int_0^{\pi/4} e^{it}\,dt \overset{\text{by Thm}}{=\!=} \frac{e^{it}}{i}\Big|_0^{\pi/4} = \frac{e^{i\frac{\pi}{4}} - 1}{i} = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)$$

def'n (1)

$$\int_0^{\pi/4} \cos t\,dt + i\int_0^{\pi/4} \sin t\,dt = [+\sin t]_0^{\frac{\pi}{4}} + i[-\cos t]_0^{\frac{\pi}{4}} \;))$$

§43 Contours

Def: (1) An <u>arc</u> C in $\mathbb{C}$ is a parametrized <u>continuous</u> curve

$$z = z(t) = x(t) + i\,y(t), \quad t \in [a,b]$$

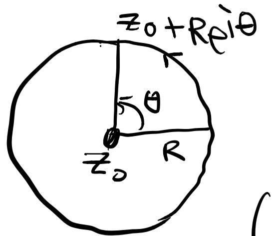
(where, $x(t)$, $y(t)$ are cts. functions over $[a,b]$)

(2) The arc C is a <u>simple arc</u> or <u>Jordan arc</u> if $z(t_1) \neq z(t_2)$, for $t_1 \neq t_2$.

(3) The arc C is <u>simple closed curve</u> or <u>Jordan curve</u> if C is simple except $z(b) = z(a)$

(4) A simple closed curve $C: z = z(t)$ is <u>positively oriented</u> when it is in the counterclockwise direction

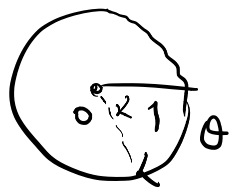
eg2: $z = z(\theta) = z_0 + Re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$



positively oriented simple closed curve.

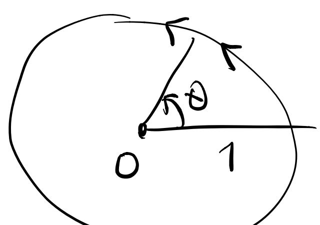
(A circle of radius R centered at z_0 in the counterclockwise direction.)

eg3: $z = e^{-i\theta}, \quad 0 \leq \theta \leq 2\pi$



"negatively" oriented circle of radius 1 centered at $z_0 = 0$.

eg4: $z = e^{2i\theta}, \quad 0 \leq \theta \leq 2\pi$



Not simple as the circle is traversed twice in the counterclockwise direction.

Def: Change of parameters (a reparametrization)

Let C be an arc parametrized by

$$z = z(t), \quad a \leq t \leq b.$$

Let $\phi : [\alpha, \beta] \to [a, b]$ be a differentiable increasing function (i.e. $\phi'(\tau) > 0, \ \forall \tau \in [\alpha, \beta]$) such that

$\phi(\alpha) = a$, $\phi(\beta) = b$ ($\Rightarrow \phi$ is 1-1 & onto)

Then $z = Z(\tau) = z \circ \phi(\tau) = z(\phi(\tau))$, $\alpha \le \tau \le \beta$

is called a <u>reparametrization</u> of $z = z(t)$, $a \le t \le b$.

And ϕ is called a <u>change of parameters</u>.

<u>Def</u>: An arc C given by $z = z(t) = x(t) + i y(t)$
$a \le t \le b$

is called a <u>differentiable arc</u> if $x'(t)$, $y'(t)$ exist and continuous on $[a, b]$.

<u>Def</u>: The <u>length</u> L of a differentiable arc

$$z = z(t) = x(t) + i y(t), \quad a \le t \le b$$

is defined by $$L = \int_a^b |z'(t)| \, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt.$$

<u>Prop</u>: The length L of a differentiable arc is independent of the parametrization.

<u>Pf</u>: Let $\phi: [\alpha, \beta] \to [a, b]$ be a change of parameters, then $L\left(\begin{matrix}\text{calculated in}\\ \text{reparametrization}\end{matrix}\right) = \int_\alpha^\beta |Z'(\tau)| \, d\tau$

$$= \int_\alpha^\beta \left| (z \circ \phi)'(\tau) \right| d\tau$$

$$= \int_\alpha^\beta \left| z'(\phi(\tau))\, \phi'(\tau) \right| d\tau$$

(as ϕ increasing, $\phi' > 0$)

$$= \int_\alpha^\beta \left| z'(\phi(\tau)) \right| \phi'(\tau)\, d\tau$$

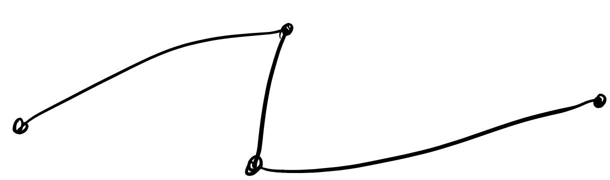
(change of variable $t = \phi(\tau)$)

$$= \int_a^b |z'(t)|\, dt = L$$

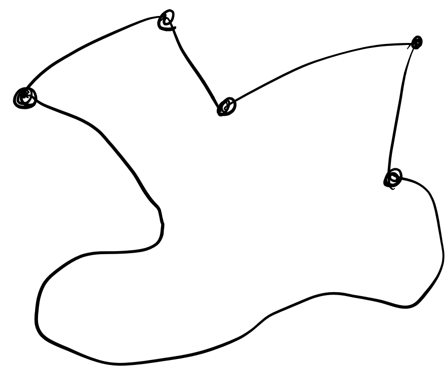
calculated in original parametrization. ※

Def: (1) An arc is called smooth (or regular) if $z(t) \in C^1[a,b]$ and $z'(t) \neq 0$, $\forall t \in [a,b]$

(2) A contour, a piecewise smooth arc, is an arc consisting of finite number of smooth arcs joining end to end.



(3) If only the initial and final points are the same, a contour is called a simple closed contour



Facts: Jordan Curve Theorem

The points on any simply closed contour C are the boundary points of 2 distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded.

(Pf: Omitted)

§44 Contour Integrals

Def: Suppose that a contour C is represented by

$$z = z(t), \quad a \leq t \leq b$$

with $z_1 = z(a)$, $z_2 = z(b)$.

(1) A cpx-valued function $f(z)$ is said to be piecewise continuous on C if $f(z(t))$ is a piecewise continuous function on $a \leq t \leq b$.

(2) The <u>contour integral</u> (a line integral) of f along C (in term of the parameter t) is

$$\boxed{\int_C f(z)\,dz \overset{\text{def}}{=} \int_a^b f(z(t))\, z'(t)\,dt}$$

(Think of $dz = z'(t)\,dt$)

<u>Note</u>: The value $\int_C f(z)dz$ is indep of the parameter t.

Pf: Let $t=\phi(\tau)$, $\alpha \le \tau \le \beta$ be a change of parameter

Then $$\int_C f(z)dz = \int_a^b f(z(t))\, z'(t)\,dt$$

$$= \int_\alpha^\beta f(z(\phi(\tau)))\, z'(\phi(\tau))\,(\phi'(\tau)d\tau)$$

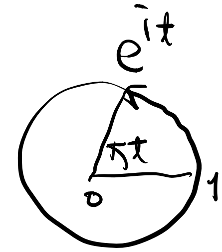
$$= \int_\alpha^\beta f(Z(\tau))\, Z'(\tau)\,d\tau$$

where $Z(\tau) = z(\phi(\tau))$ ※

Def: Let C be a contour represented by $z(t)$, $a \le t \le b$

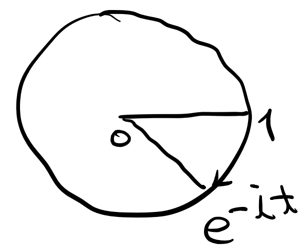
Then $\underline{\underline{-C}}$ is the contour defined by the reparametrization $z = z(-t)$, $-b \le t \le -a$

(Same set, but opposite direction)



eg: $C : z = e^{it}, \quad 0 \le t \le 2\pi$

$-C : z = e^{-it}, \quad -2\pi \le t \le -0 = 0$



To see this more clearly, we reparametrize $-C$ by

$$\tau = t + 2\pi \quad (\Leftrightarrow t = \tau - 2\pi]$$

$$\phi : [0, 2\pi] \to [-2\pi, 0]$$
$$\tau \longmapsto \phi(\tau) = \tau - 2\pi$$

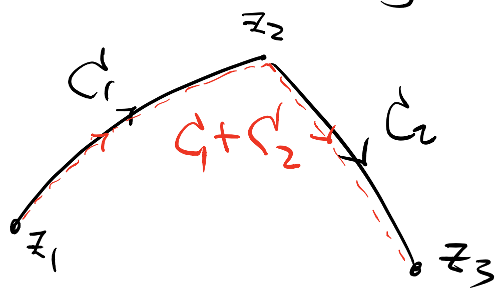
reparametrization (for $-C$)

$$z(\phi(\tau)) = e^{-it}$$
$$= e^{-i(\tau - 2\pi)}$$
$$= e^{-i\tau} e^{2\pi i}$$
$$= e^{-i\tau}, \quad 0 \le \tau \le 2\pi .$$

Def: (1) If C_1 is a contour from z_1 to z_2 &

C_2 is a contour from z_2 to z_3.

Then sum $C = C_1 + C_2$ is the contour from z_1 to z_3 by first travel from z_1 to z_2 along C_1 and then z_2 to z_3 along C_2.

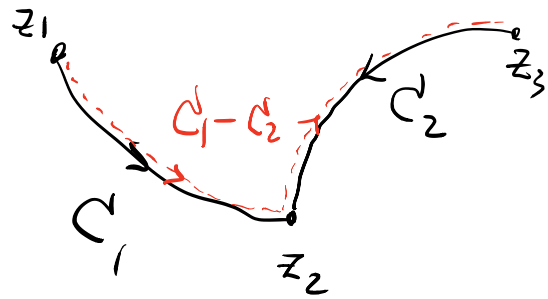


(And we can define $C_1 + C_2 + \cdots + C_N$ similarly.)

(2) If C_1 is a contour from z_1 to z_2, &

C_2 is a contour from z_3 to z_2.

Then $C_1 + (-C_2)$ is well-defined as in (1) & is denoted by $C_1 - C_2$



Properties

(1) $\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$, for const. z_0

(2) $\int_C [f(z) \pm g(z)]\,dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz$

(3) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

(4) $\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

Pf: (1) & (2) are easy.

(3) Let $C: z = z(t)$, $a \le t \le b$.

Then $-C: z = z(-t)$, $-b \le t \le -a$

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f(z(-t))\left[\frac{d}{dt} z(-t)\right] dt$$

$$= \int_{-b}^{-a} f(z(-t))\left(-z'(-t)\right) dt$$

$$= \int_{-b}^{-a} f(z(-t))\, z'(-t)\, d(-t)$$

(change of variable) $= \int_b^a f(z(t))\, z'(t)\,dt$

$$= -\int_a^b f(z(t))z'(t)dt$$

$$= -\int_C f(z)dz \quad ※$$

(4) Let $C_1: \ z = z_1(t), \ a \le t \le c$

$C_2: \ z = z_2(t), \ c \le t \le b$

(by suitable change of parameters)

Then $C_1 + C_2: \ z = \begin{cases} z_1(t), & a \le t \le c \\ z_2(t), & c \le t \le b \end{cases}$

$$\Rightarrow \int_{C_1+C_2} f(z)dz = \int_a^b f(z(t))z'(t)dt$$

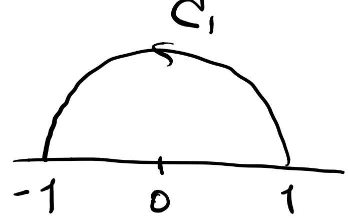
$$= \int_a^c f(z(t))z'(t)dt + \int_c^b f(z(t))z'(t)dt$$

$$= \int_a^c f(z_1(t))z_1'(t)dt + \int_c^b f(z_2(t))z_2'(t)dt$$

$$= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \quad ※$$

§45 Some examples

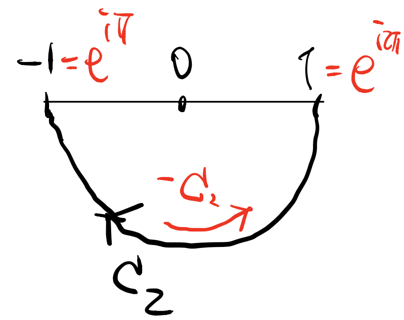
eg 1 (a) Evaluate $\int_{C_1} \frac{dz}{z}$ along $C_1: z = e^{i\theta},\ 0 \le \theta \le \pi$



Soln: $\int_{C_1} \frac{dz}{z} = \int_0^{\pi} \frac{d(e^{i\theta})}{e^{i\theta}}$

$$= \int_0^{\pi} \frac{i e^{i\theta} d\theta}{e^{i\theta}} = i \int_0^{\pi} d\theta = \pi i .$$

(b) Evaluate $\int_{-C_2} \frac{dz}{z}$ along $-C_2$



Soln: Parametrize $-C_2: z = e^{i\theta}$, $\pi \le \theta \le 2\pi$

$$\Rightarrow \int_{-C_2} \frac{dz}{z} = \int_{\pi}^{2\pi} \frac{d(e^{i\theta})}{e^{i\theta}} = \int_{\pi}^{2\pi} i\, d\theta = \pi i$$

Notes (1) By Thm, $\int_{C_2} \frac{dz}{z} = -\int_{-C_2} \frac{dz}{z} = -\pi i$

(2) C_1 & C_2 have the same beginning & end points (1 to −1), but $\int_{C_1} \frac{dz}{z} \neq \int_{C_2} \frac{dz}{z}$.

$\therefore$ Contour integrals depends on contour, not just beginning & end points.