egs (1) Find $\lim_{z\to -1} \frac{iz+3}{z+1} = \lim_{z\to -1} f(z)$, $f(z)=\frac{iz+3}{z+1}$

Soln: Consider $\lim_{z\to -1} \frac{1}{f(z)} = \lim_{z\to -1} \frac{1}{\left(\frac{iz+3}{z+1}\right)}$

$$= \lim_{z\to -1} \frac{z+1}{iz+3}$$

$$= 0$$

$\therefore \lim_{z\to -1} \frac{iz+3}{z+1} = \infty$. according to the Thm (Def) #

(2) Find $\lim_{z\to\infty} \frac{2z+i}{z+1}$

Soln: Consider $(\zeta = \frac{1}{z})$, $\lim_{\zeta\to 0} \frac{\frac{2}{\zeta}+i}{\frac{1}{\zeta}+1}$

$$= \lim_{\zeta\to 0} \frac{2+i\zeta}{1+\zeta} = 2$$

$\therefore \lim_{z\to\infty} \frac{2z+i}{z+1} = 2$. #

(3) Find $\lim_{z\to\infty} \frac{2z^3-1}{z^2+1}$

Soln: Consider $\lim_{z\to 0} \dfrac{1}{\left(\dfrac{2\left(\frac{1}{z}\right)^3-1}{\left(\frac{1}{z}\right)^2+1}\right)}$

$$= \lim_{z\to 0} \frac{z(1+z^2)}{2-z^3} = 0$$

$\therefore$ By Thm(Def), $\lim_{z\to\infty} \dfrac{2z^3-1}{z^2+1} = \infty$. ##

§18 Continuity

Def: A function f is continuous at a point z_0

if $\lim_{z \to z_0} f(z)$ exists, $f(z_0)$ exists

and $\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$

Since "limits" are the same as in functions of 2-variables, we have

Thm3 If $f(z) = u(x,y) + i v(x,y)$

Then $u(x,y)$, $v(x,y)$ are continuous at (x_0, y_0)

$\Leftrightarrow$ f is continuous at $z_0 = x_0 + i y_0$

(Pf: Omitted)

Thm1: Composition of continuous functions is continuous.

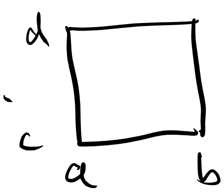
Thm2: If f is continuous at z_0 & $f(z_0) \neq 0$, then $f(z) \neq 0$, $\forall z \in B_\varepsilon(z_0)$ for some $\varepsilon > 0$.

Thm 4 If f is continuous on a region R that is both "closed" and bounded, then

$$\exists\, M > 0 \text{ s.t. } |f(z)| \le M, \ \forall z \in R.$$

where "equality" holds at least for one point.

("closed" means "R includes its own boundary"

egs. 

$R = \{ a \le x \le b \ \& \ c \le x \le d \}$,

and 

$\overline{B_r(z_0)} = \{ z : |z - z_0| \le r \}$)

§19 (Complex) Derivatives

Def: Let f be a (cpx-valued) function whose domain of definition contains a neighborhood $B_\varepsilon(z_0) = \{|z-z_0| < \varepsilon\}$ of a point z_0. The (complex) derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and the function f is said to be differentiable at z_0 when $f'(z_0)$ exists.

Usual notation

$$\begin{cases} \Delta z = z - z_0 \\ \Delta w = f(z) - f(z_0) = f(z_0 + \Delta z) - f(z_0), \end{cases}$$

We often drop the subscript on z_0 and write

$$\Delta w = f(z + \Delta z) - f(z).$$

Then $$\boxed{f'(z) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}}$$

eg 1 For $w = f(z) = \frac{1}{z}$, we have

$$\frac{dw}{dz} = -\frac{1}{z^2} \quad \text{or} \quad f'(z) = -\frac{1}{z^2}$$

(Ex: Same proof for $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ works here)

eg2: If $w = f(z) = \bar{z} = x - iy$

(i.e. $u = x$, $v = -y$. They are differentiable.)

Then $$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \frac{\overline{(\Delta z)}}{\Delta z} \text{ has no limit as } \Delta z \to 0.$$

$$\left(= \begin{cases} 1 & \text{along } \Delta z = \Delta x + i0 \\ -1 & \text{along } \Delta z = 0 + i\Delta y \end{cases} \right)$$

$\therefore$ $\bar{z}$ is not (cpx) differentiable !

eg3: $f(z) = |z|^2 = x^2 + y^2 = z\bar{z}$

is not differentiable unless $z = 0$.

(By eg2)

Notes: (1) eg3 $\Rightarrow$ function can be (cpx) differentiable at one point but non-differentiable elsewhere.

(2) $\bar{z}$, $|z|^2$ are differentiable as real variable function $\left(\begin{array}{l} u = x \\ v = y \end{array} \ ; \ \begin{array}{l} u = x^2 + y^2 \\ v = 0 \end{array} \right)$

but not cpx differentiable.

(3) As in real case, continuity $\not\Rightarrow$ differentiability but differentiability $\Rightarrow$ continuity.