Pf (of the Fundamental Theorem of Möbius Geometry)

The formula

$$\frac{w-w_2}{w-w_3} \cdot \frac{w_1-w_3}{w_1-w_2} = \frac{z-z_2}{z-z_3} \cdot \frac{z_1-z_3}{z_1-z_2}$$

provided in the last lecture can be seen from the following steps:

Step 1: $\forall z_1, z_2, z_3$ distinct (extended) cpx numbers, $\exists$ Möbius transformation T such that

$$Tz_1 = 1, \quad Tz_2 = 0, \quad Tz_3 = \infty \qquad \text{— } (*)$$

Pf of Step 1:

$$\left.\begin{aligned} Tz_2 = 0 &\Rightarrow \frac{*(z-z_2)}{***} \\ Tz_3 = \infty &\Rightarrow \frac{***}{*(z-z_3)} \end{aligned}\right\}$$

$\Rightarrow$ Tz must be the form

$$Tz = \beta \frac{z-z_2}{z-z_3}$$ for some complex number β.

Then $Tz_1 = 1 \Rightarrow \quad 1 = \beta \cdot \frac{z_1-z_2}{z_1-z_3}$

$$\Rightarrow \quad \beta = \frac{z_1 - z_3}{z_1 - z_2}$$

Hence $$Tz = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$ #

Step 2 :

$$\begin{tikzcd} z_1 \arrow[r, "T"] \arrow[rr, bend right, "U^{-1} \circ T"'] & 1 & w_1 \arrow[l, "U"'] \\ z_2 \arrow[r] & 0 & w_2 \arrow[l] \\ z_3 \arrow[r] & \infty & w_3 \arrow[l] \end{tikzcd}$$

$\forall$ distinct z_1, z_2, z_3 & distinct w_1, w_2, w_3,

By step 1, $\exists$ T s.t. $\begin{cases} Tz_1 = 1 \\ Tz_2 = 0 \\ Tz_3 = \infty \end{cases}$

and U s.t. $\begin{cases} Uw_1 = 1 \\ Uw_2 = 0 \\ Uw_3 = \infty \end{cases}$

(T, U are Möbius transformations)

Then $S = U^{-1} \circ T$ is a Möbius Transformation

such that $S z_1 = U^{-1} \circ T(z_1) = U^{-1}(T z_1)$

$$= U^{-1}(1) = w_1$$

Similarly $S z_2 = w_2$ & $S z_3 = w_3$.

Hence, we've proved that for any distinct z_1, z_2, z_3 and distinct w_1, w_2, w_3, $\exists$ a Möbius transformation S such that $S z_i = w_i$, $i = 1, 2, 3$. #

(Ex: Why this gives the formula?)

<u>Finally</u> (<u>Uniqueness</u>):

If U_1 & U_2 are Möbius transformations s.t.

$$U_k(z_i) = w_i, \quad i = 1, 2, 3; \quad k = 1, 2$$

Then $U_1 = U_2$.

<u>Pf of Final step</u>:

Consider $U_2^{-1} \circ U_1$, then it is a Möbius transformation such that $U_2^{-1} \circ U_1(z_i) = U_2^{-1}(w_i) = z_i$, $i = 1, 2, 3$.

$\Rightarrow$ $T_2^{-1} \circ T_1$ has at least 3 fixed points

(as z_1, z_2, z_3 are distinct.)

By lemma (of the last lecture),

$$T_2^{-1} \circ T_1 = Id_{\hat{\mathbb{C}}}$$

$$\Rightarrow \quad T_1 = T_2 \ . \quad ※$$

Corollary: All figures consisting of 3 distinct points are congruent in Möbius geometry.

Remark: This corollary $\Rightarrow$ Möbius geometry is not isomorphic to Euclidean geometry, and Euclidean distance is not an invariant.

Invariants of Möbius Geometry

- Angle measurement

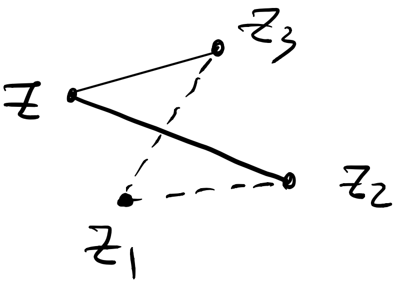
  Möbius transformations are conformal

  $\Rightarrow$ (Euclidean) angle measure is an invariant of Möbius Geometry.

- **Cross Ratio**

> **Def:** The cross ratio is the following function of 4 (extended) complex variables:
>
> $$(z, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}.$$

Remarks (1)



$$\frac{z-z_2}{z-z_3} \Big/ \frac{z_1-z_2}{z_1-z_3}$$

(2) If z_1, z_2, z_3 are held constants, then as a function of z, $Tz = (z, z_1, z_2, z_3)$ is the unique Möbius transformation sending z_1 to 1, z_2 to 0, and z_3 to ∞.

> **Thm:** Let z, z_1, z_2, z_3 be 4 distinct points on $\hat{\mathbb{C}}$, then $\forall\ S \in \mathbb{M}$,
>
> $$(Sz, Sz_1, Sz_2, Sz_3) = (z, z_1, z_2, z_3)$$

Pf: By the remark (2) above,

$Tz = (z, z_1, z_2, z_3)$ is the unique Möbius transformation such that

$$Tz_1 = 1, \quad Tz_2 = 0, \quad Tz_3 = \infty.$$

Consider the composition $T \circ S^{-1} \in M$

Note that
$$\begin{cases} T \circ S^{-1}(Sz_1) = Tz_1 = 1 \\ T \circ S^{-1}(Sz_2) = Tz_2 = 0 \\ T \circ S^{-1}(Sz_3) = Tz_3 = \infty \end{cases}$$

$$\Rightarrow \quad T \circ S^{-1}(z) = (z, Sz_1, Sz_2, Sz_3) \quad (\forall z)$$

Therefore

$$\begin{aligned} Tz = T \circ S^{-1}(Sz) &= (Sz, Sz_1, Sz_2, Sz_3) \\ \| \quad & \\ (z, z_1, z_2, z_3) & \end{aligned}$$

※

Thm: The cross ratio (z, z_1, z_2, z_3) is <u>real</u> if and only if the 4 points lie on a <u>Euclidean circle</u> or <u>straight line</u>.

Pf: $(z, z_1, z_2, z_3) \in \mathbb{R}$

$\Leftrightarrow (Tz, Tz_1, Tz_2, Tz_3) \in \mathbb{R}, \ \forall T \in \mathbb{M}$.

Let $T \in \mathbb{M}$ be the Möbius transformation such that $Tz_1 = 1, Tz_2 = 0, Tz_3 = -1$.

Then

$$\mathbb{R} \ni (z, z_1, z_2, z_3) = (Tz, 1, 0, -1)$$

$$= \frac{Tz - 0}{Tz - (-1)} \cdot \frac{1 - (-1)}{1 - 0}$$

$$= \frac{2Tz}{1 + Tz}$$

If $(z, z_1, z_2, z_3) = 2$, then $Tz = \infty$

If $(z, z_1, z_2, z_3) \neq 2$, then $Tz = \dfrac{(z, z_1, z_2, z_3)}{2 - (z, z_1, z_2, z_3)} \in \mathbb{R}$

In any case, Tz, Tz_1, Tz_2, Tz_3 lie on the x-axis

Therefore, z, z_1, z_2, z_3 lie on a Euclidean circle or a straight line (since Möbius transforms maps lines/circles to lines/circles.)

#

Clines

Def: A subset C of the complex plane is a <u>cline</u> if C is a Euclidean circle or Euclidean straight line.

Thm: If C is a cline, then $T(C)$ is a cline, $\forall T \in \mathbb{M}$.

(Pf: Ex!)

Remark: All <u>circles and straight lines are congruent</u> to each other in Möbius geometry: (i) circle determined by 3 points

(ii) straight line is just a "circle" passing throught ∞.

(Ex!)

Symmetry

Def: Let C be a cline passing through 3 distinct points $z_1, z_2, \& z_3$. Two points z and z^* are called symmetric with respect to C if

cpx conjugate

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

eg.: If z_1, z_2, z_3 are 3 distinct points on x-axis, then

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

$$= \overline{\left(\frac{z-z_2}{z-z_3} \cdot \frac{z_1-z_3}{z_1-z_2}\right)}$$

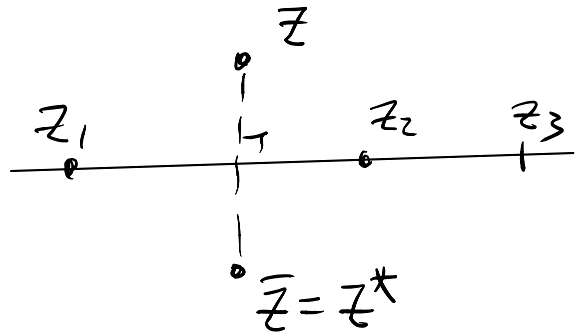
$$= \frac{\bar z - \bar z_2}{\bar z - \bar z_3} \cdot \frac{\bar z_1 - \bar z_3}{\bar z_1 - \bar z_2}$$

$$= (\bar z, \bar z_1, \bar z_2, \bar z_3)$$

$$= (\bar z, z_1, z_2, z_3) \quad \left(\text{since } z_i \text{ are real}\right)$$

$\Rightarrow z^* = \bar z$ (since $Tz = (z, z_1, z_2, z_3)$ is invertible)

which is the usual mirror symmetry of z across the x-axis



Remarks: (i) In this case, we see that one can take any 3 points on the x-axis to give the symmetry wrt x-axis. Similarly, this is true for any cline C.

(ii) z, z^* symmetric wrt C

$\Leftrightarrow Tz, Tz^*$ symmetric wrt $T(C)$ (Ex!)

ie.

$$\boxed{T(z^*) = (Tz)^*}$$

wrt C wrt $T(C)$

eg: If $C = \{z : |z-a|^2 = R^2\}$, and $z_1, z_2, z_3 \in C$

Then z, z^* symmetric wrt C

$$\Leftrightarrow (z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

(Cross-ratio is invariant under Möbius transformations)

$$= \overline{(z-a, z_1-a, z_2-a, z_3-a)}$$

$$= (\overline{z-a}, \overline{z_1-a}, \overline{z_2-a}, \overline{z_3-a})$$

($z_1, z_2, z_3 \in C$)

$$= \left(\bar{z}-\bar{a}, \frac{R^2}{z_1-a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}\right)$$

$$= \left(\frac{\bar{z}-\bar{a}}{R^2}, \frac{1}{z_1-a}, \frac{1}{z_2-a}, \frac{1}{z_3-a}\right)$$

$$= \left(\frac{R^2}{\bar{z}-\bar{a}}, z_1-a, z_2-a, z_3-a\right)$$

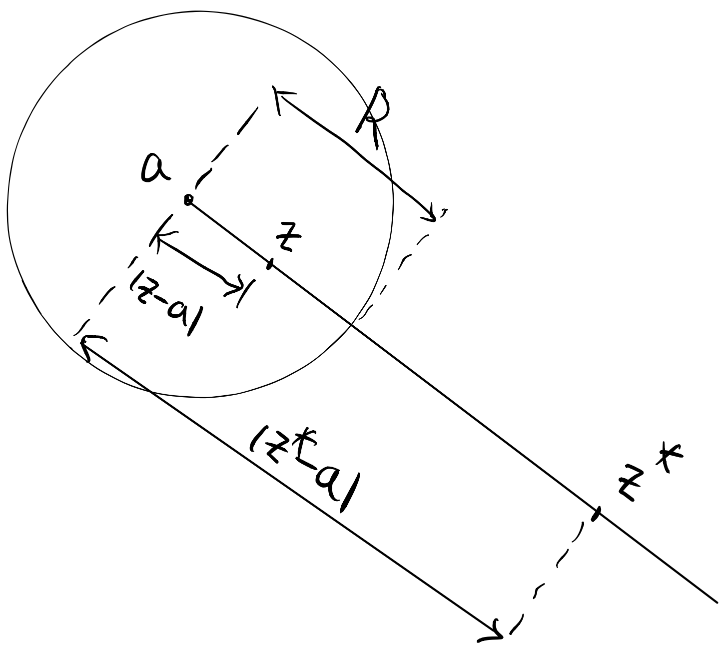
$$= \left(\frac{R^2}{\bar{z}-\bar{a}} + a, z_1, z_2, z_3\right)$$

$$\Longleftrightarrow \quad \boxed{z^* = \frac{R^2}{\bar{z}-\bar{a}} + a}$$

or $$\boxed{z^* - a = \frac{R^2}{|z-a|^2}(z-a)}$$

which implies $$\boxed{|z^*-a||z-a| = R^2}$$

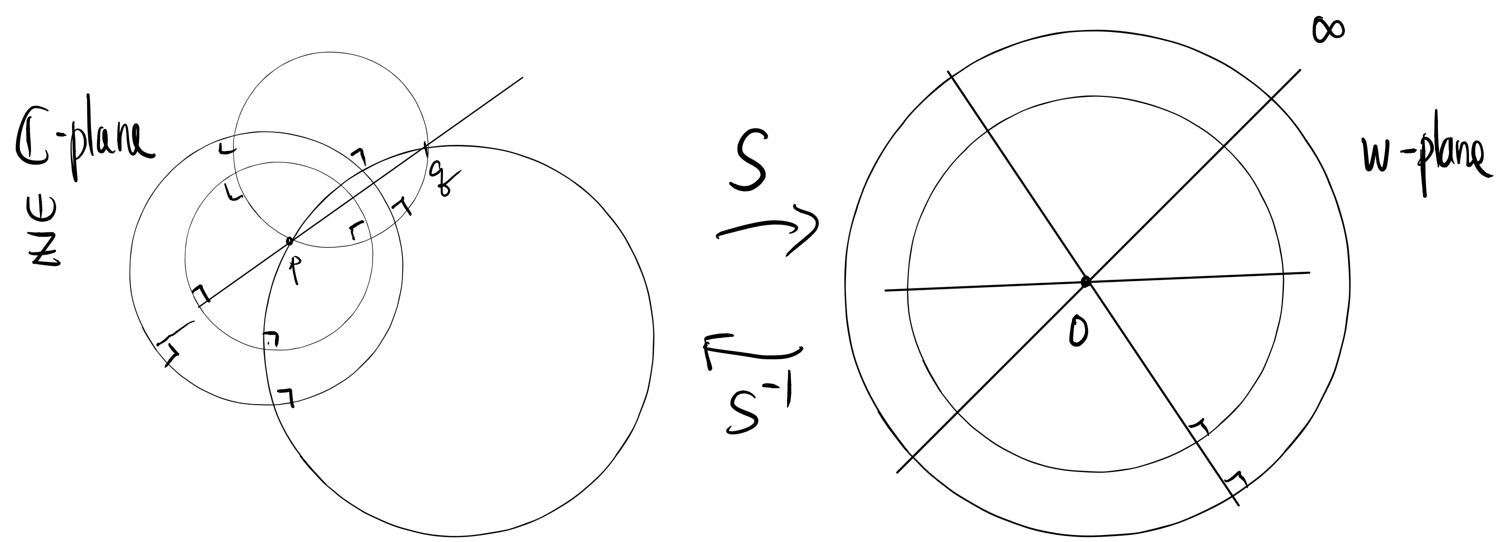
and $$\boxed{(z^*)^* = z}$$ (Ex!)



Ch6 Steiner Circles

Families of clines

Let $p, q \in \mathbb{C}$, the family of all clines passing through p and q is called the Steiner circles of the first kind with respect to points p and q



Consider the transformation

$$w = Sz = \frac{z-p}{z-q}$$

Then

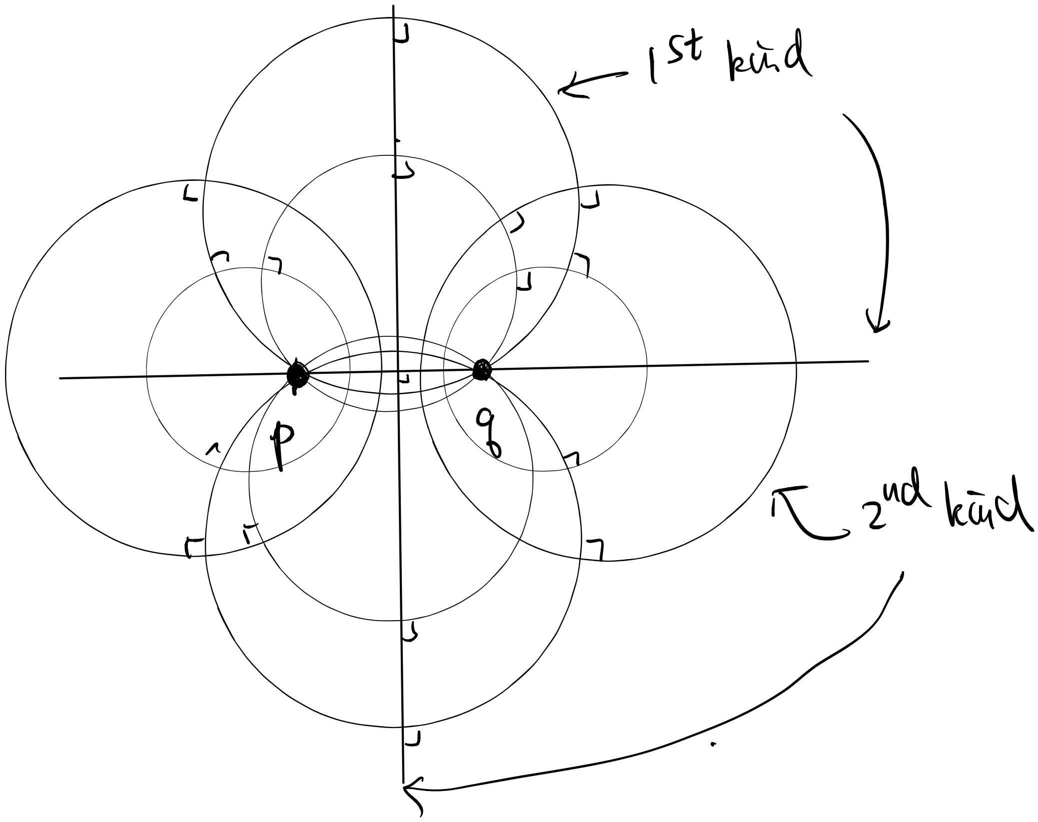
$$\begin{cases} p \xmapsto{S} 0 & (\text{i.e. } Sp = 0) \\ q \xmapsto{S} \infty & (\text{i.e. } Sq = \infty) \end{cases}$$

Recall that Möbius transformations takes clines to clines,
the image of the circles (clines) in the Steiner circles of the 1^{st} kind wrt p & q form the "Steiner circles" of the 1^{st} kind wrt 0 & ∞.

In the w-plane, it is easy to see that there is another family of clines orthogonal to the "Steiner circles" of the 1^{st} kind: namely the family of circles centered at $w=0$.

The pull back of these circles $\{|w|=k\}$ in the w-plane by S^{-1} form a family of clines on z-plane which is called the Steiner circles of the second kind (wrt p & q)

(also called circles of Apollonius)



By definition, the Steiner circles of 2nd kind (wrt p & q) have the equation:

$$\boxed{\frac{|z-p|}{|z-q|} = k}$$

<u>Remark</u>: The families of Steiner circles of 1st & 2nd kinds can be regarded as a generalization of <u>polar coordinates to Möbius geometry</u>.