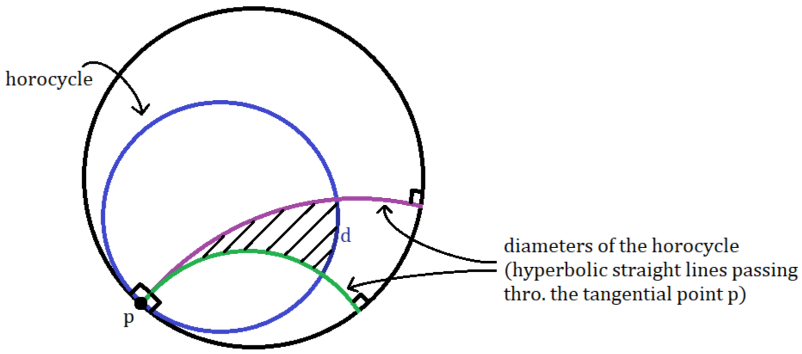
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5120 Topics in Geometry 2018/19
Classwork 3 Question, solution and discussions

Question:

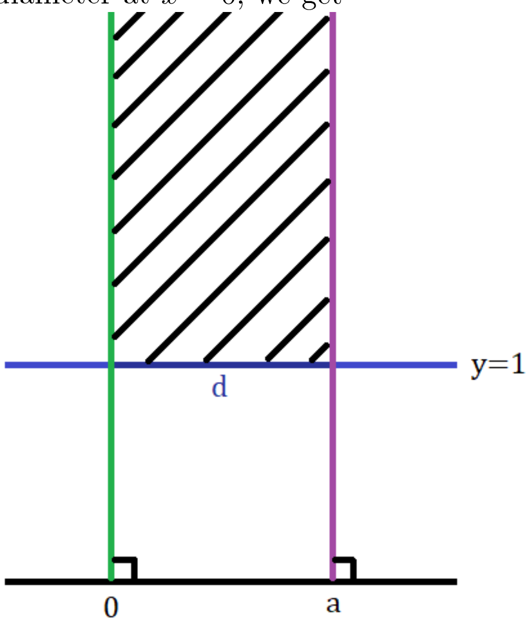
1.



Suppose that the hyperbolic length of the arc on the horocycle between two diameter is d. Find the shaded area shown in the figure in terms of d.
(Hint: transform it to upper half plane model)

Solution:

1. Transform it to upper half plane model. Putting p at ∞, the horocycle at $y = 1$, and the green diameter at $x = 0$, we get



for some positive real number a.
As the hyperbolic length of the arc is d, we have $d = \int_0^a \frac{1}{1^2} dx = a$.
So the shaded area is $\int_0^d \int_1^\infty \frac{dydx}{y^2} = d[-\frac{1}{y}]_1^\infty = d$.

Discussions:
Unfortunately, nobody had managed to do the transformation properly. Let me explain the solution in detail.
I have used different colours to represent different objects, for helping you to keep track on the correspondence between the objects in the figure in the question and the objects in the figure in the solution.

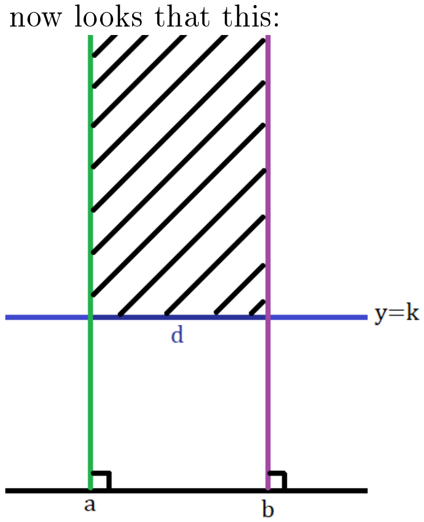
A general tip for hyperbolic geometry is that if there is a special point (a point with lots of things passing through it) in the interior, it's usually a good idea to use the disk model and put the point at the centre. If there is a special point in the boundary (an ideal point), it's usually a good idea to use the upper half plane model and put the point at ∞.
In this question, a lot of things pass through p, so using the upper half plane model and put p at ∞ will simplify things a lot.

Now we try to draw the diagram in the upper half plane model, with p being put at ∞. The boundary of the disk in the disk model corresponds to the real line (including the point ∞) in the upper half plane model, so the black circle now becomes the real line. The green and purple curves are hyperbolic lines passing through p (now at ∞), so now they must be vertical lines. The blue horocycle is a cline that passes through p (now at ∞), so now it must be a straight line (a cline that passes through ∞ is a straight line). Also, the horocycle touches the black circle **only** at p (now at ∞), so now it must be a straight line parallel to the real line, which means that it is a horizontal line.

Putting the horocycle at $y = 1$ and the green line at $x = 0$ is not necessary. I did that because it simplifies the calculations a bit. I am allowed to do that because there are hyperbolic transformations (in the upper half plane model) moving horizontal lines up and down ($T(z) = \lambda z$, $\lambda \in \mathbb{R}$), and moving everything left and right ($T(z) = z + \lambda$, $\lambda \in \mathbb{R}$).

Don't worry if you don't understand the "put the horocycle at $y = 1$ and the green line at $x = 0$" bit. We will now do the question again, without doing that.

After the transformation to the upper half plane model, with p being put at ∞, the diagram now looks that this:



Note that the d there indicates the hyperbolic length of the line segment, not the Euclidean length.

By defintion of hyperbolic area, the shaded area is $\int_a^b \int_k^\infty \frac{dydx}{y^2} = (b-a)[-\frac{1}{y}]_k^\infty = \frac{b-a}{k}$.

How to write $\frac{b-a}{k}$ in terms of d? Well, let's try to use the information that the hyperbolic length of the line segment is d.

Parametrize the line segment by $z(t) = t + ki$, from $t = a$ to $t = b$.

$d =$ hyperbolic length of the line segment $= \int_a^b \frac{|z'(t)|}{y(t)} dt = \int_a^b \frac{1}{k} dt = \frac{b-a}{k}$.

Oh, great. So the shaded area is just d. ☺

This question is probably quite difficult for people who are not familiar with doing non-Euclidean geometry, because it involves actually doing geometry instead of just solving things algebraically. However, this question is straightforward, in the sense that all the steps are "standard" and you don't need to come up with any clever ideas to do it.

Many things go through p, so it's a standard move to use the upper half plane model and put p at ∞ (so more clines will become straight lines, which are easier than circles to deal with). It's given that the hyperbolic length of the arc is d, so obviously we shall substitute everything into the formula for hyperbolic length. We need to find the hyperbolic area, so obviously we shall substitute everything into the formula for hyperbolic area.

Some students did the transformation to the upper half plane correctly, but didn't put p at ∞. Some students put p at ∞, but did the transformation for the horocycle wrongly. As a practice for doing transformations, I suggest you to have a look at Solution 2 of Question 5 in "Solutions to homework 1" on the course webpage. And remember the hints for turning more clines into straight lines and make things easier:
If a lot of things pass through an ideal point, use upper half plane model and put the point at ∞.
If a lot of things pass through a point in the hyperbolic plane (not ideal point), use the disk model and put the point at the origin.