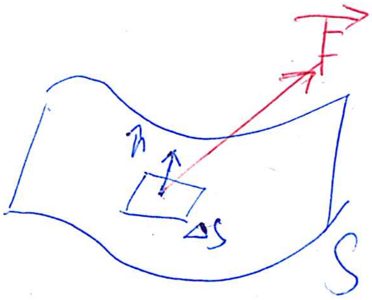
Surface Integrals

$\iint_S f\,dS$ or. $\iint_S \vec{F}\cdot\hat{n}\,dS$.



Computation:

Find dS and $d\vec{S} = \hat{n}\cdot dS$.

(1). When S is parametrized: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

$$\boxed{\hat{n}\,dS = \pm\left(\vec{r_u}\times\vec{r_v}\right)du\,dv}$$

$$\boxed{dS = \|\vec{r_u}\times\vec{r_v}\|\,du\,dv}$$

Example: graph: $\vec{r}(x,y) = \langle x, y, f(x,y)\rangle$

$r_x = \langle 1, 0, f_x\rangle$, $r_y = \langle 0, 1, f_y\rangle$

$$\Rightarrow \boxed{\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\,dx\,dy}$$

(2). When S is given by an equation. $g(x,y,z) = 0$

$\vec{N} = \nabla g$. and $\hat{n} = \pm\dfrac{\vec{N}}{|\vec{N}|}$.

$$\boxed{dS = \frac{|\vec{N}|}{\vec{N}\cdot\hat{k}}\,dx\,dy}$$

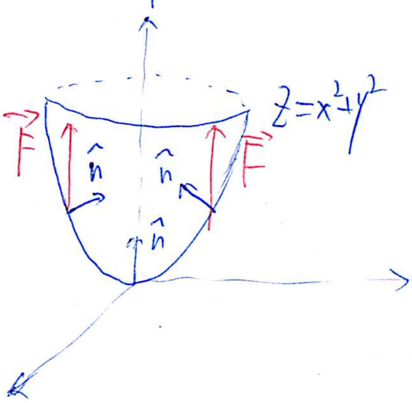
$$\boxed{\hat{n}\cdot dS = \pm\frac{\vec{N}}{\vec{N}\cdot\hat{k}}\,dx\,dy}$$

Example: graph: $g(x,y,z) = z - f(x,y) = 0$

$\vec{N} = \nabla g = \langle -f_x, -f_y, 1\rangle$.

$$\Rightarrow \boxed{\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\,dx\,dy}$$

Example: $\vec{F} = z\hat{k}$. S = portion of paraboloid $z = x^2+y^2$ above unit disk, $\hat{n}$ upwards.



- $\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \langle 0,0,z\rangle\cdot\langle -2x, -2y, 1\rangle\,dx\,dy$

  $= \iint_S z\,dx\,dy$

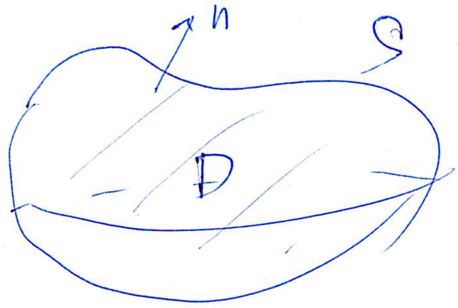
  $= \iint (x^2+y^2)\,dx\,dy$

  $= \int_0^{2\pi}\int_0^1 r^2\cdot r\,dr\,d\theta = \frac{\pi}{2}$

- $\text{Area}(S) = \iint_S dS = \iint \sqrt{1+4x^2+4y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1\sqrt{1+4r^2}\,r\,dr\,d\theta$

  $= \frac{\pi}{6}(5\sqrt{5}-1)$.

Divergence Theorem



S closed surface enclosing D

oriented with $\hat{n}$ outwards

$\vec{F}$ defined everywhere in D.

Then,
$$\boxed{\oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV.}$$

Physical intuition: the amount of fluid generated by sources in D = the amount of fluid flows out of S.

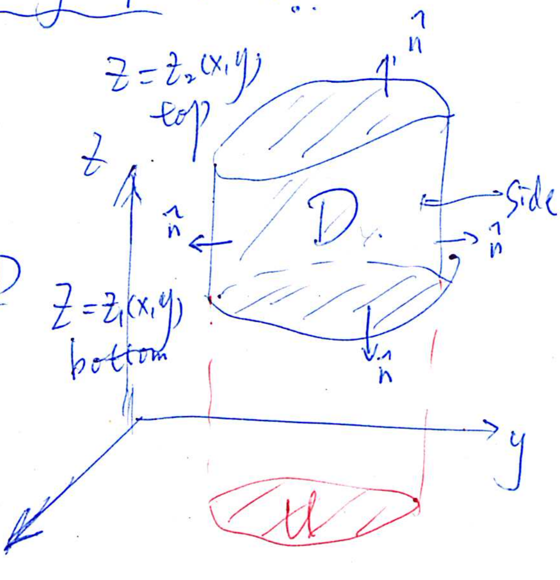
where $\boxed{\operatorname{div}\langle P, Q, R\rangle = P_x + Q_y + R_z}$

Proof: (1) Simplify integrand: only need to prove $\oiint_S \langle 0,0,R\rangle\cdot\hat{n}\,dS = \iiint_D R_z\,dV$

(then get the general case by summing three such identities, one for each component)

(2) Simplify domain: decompose D into vertically simple domains:

Prove the identity for each small pieces, and sum together to get the general case.



(3) Main part: $\vec{F} = \langle 0,0,R\rangle$ on vertically simple D

$$-\iiint_D R_z\,dV = \iint_U \left(\int_{z_1(x,y)}^{z_2(x,y)} R_z\,dz\right)dx\,dy$$

$$= \boxed{\iint_U \left(R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\right)dx\,dy}$$

$$\| $$

$$-\oiint_S = \iint_{top} + \iint_{bottom} + \iint_{sides}: = \boxed{\iint_U R(x,y,z_2(x,y))\,dx\,dy - \iint_U R(x,y,z_1(x,y))\,dx\,dy}$$

$$\iint_{top} = \iint_{top} \langle 0,0,R\rangle\cdot\left\langle -\frac{\partial z_2}{\partial x}, -\frac{\partial z_2}{\partial y}, 1\right\rangle dx\,dy = \iint_U R(x,y,z_2(x,y))\,dx\,dy$$

$$\iint_{bottom} = \iint_{bottom} \langle 0,0,R\rangle\cdot\left\langle +\frac{\partial z_1}{\partial x}, +\frac{\partial z_1}{\partial y}, -1\right\rangle dx\,dy = -\iint_U R(x,y,z_1(x,y))\,dx\,dy$$

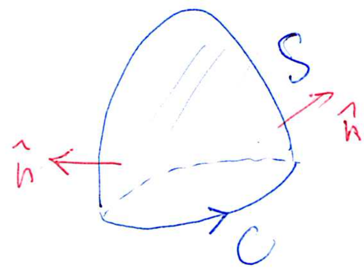
$\iint_{sides} = 0$ since $\langle 0,0,R\rangle$ is tangent to sides. $\square$

∇ notation:

∇ "del" $= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$.

- $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$. <u>gradient</u>.
- $\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ <u>divergence</u>.
- $\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ <u>Laplacian</u>.
- $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$. <u>curl</u>

Stokes' theorem



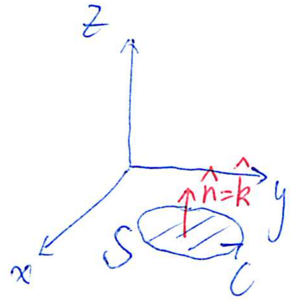
$\hat{n}$. Compatible orientation with C.

C <u>closed</u> curve

S <u>any</u> surface bounded by C.

$\vec{F}$ defined everywhere on S.

Then $$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot \hat{n}\, dS}$$

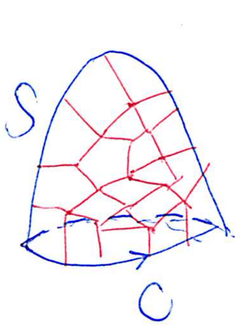


<u>Idea of proof</u>: – True for C, S in xy-plane

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy \underset{\substack{\uparrow \\ \text{(Green's thm)}}}{=} \iint_S (Q_x - P_y)\,dx\,dy = \iint_S (\nabla \times \vec{F}) \cdot \underbrace{\hat{n}\,dS}_{= \hat{k}\,dx\,dy}$$

– True for C, S in <u>any</u> plane,

because work, flux, curl make sense <u>indep.</u> of coordinate system.



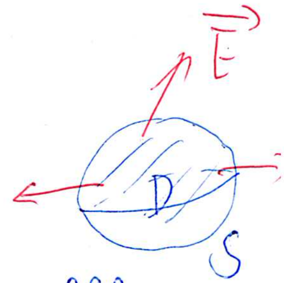
– Given any S, decompose it into tiny, almost flat pieces.

Sum of flux through each piece = total flux through S

Sum of work around each piece = total work along C.

Physical Applications

1. Maxwell's equation

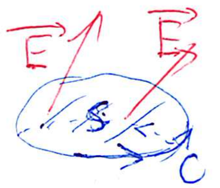


- Gauss-Columb law: $\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$ (ρ: electric charge density; ε_0: const)

$\xLeftrightarrow{\text{divergence thm}}$ $\displaystyle\oiint_S \vec{E}\cdot\hat{n}\,dS = \iiint_D \nabla\cdot\vec{E}\,dV$ ($\oiint_S \vec{E}\cdot\hat{n}\,dS$: Flux of E out of S.)

$$= \frac{1}{\varepsilon_0}\iiint \rho\, dV$$

$$= \frac{Q}{\varepsilon_0}. \quad \leftarrow \text{electric charge in } D.$$



- Faraday's law: $\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$ $\xLeftrightarrow{\text{Stokes thm}}$

$$\oint_C \vec{E}\,d\vec{r} = \iint_S (\nabla\times\vec{E})\,d\vec{S}$$

("Voltage": $\oint_C \vec{E}\,d\vec{r}$)

$$= \iint_S -\frac{\partial \vec{B}}{\partial t}\,d\vec{S}$$

$$= -\frac{d}{dt}\underbrace{\iint_S \vec{B}\cdot d\vec{S}}_{\text{flux of magnetic field.}}$$

- Gauss law: $\boxed{\nabla\cdot\vec{B} = 0}$.

- Ampère's law: $\boxed{\nabla\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}}$ ($\varepsilon_0\mu_0$: const; $\vec{J}$: vector current density.)

2. Diffusion/Heat equation.

u = concentration/temperature at a given pt at a given moment
$= u(x,y,z,t)$.

Satisfies:

$$\boxed{\frac{\partial u}{\partial t} = k\nabla^2 u = k\nabla\cdot\nabla u = k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)}$$

(k: const.)

Why this equation? ~~Let~~ Flow of smoke $= \vec{F}$.

- Smoke flow from high concentration to low concentration.

  $\vec{F}$ direct along $-\nabla u$.

  In fact, reasonable to assume $\boxed{\vec{F} = -k\nabla u}$ for some const k.

- Relate $\vec{F}$ and $\frac{\partial u}{\partial t}$ using divergence thm.

Flow out of D through S is

$$\iiint_D \operatorname{div}\vec{F}\, dV \underset{\substack{\text{divergence}\\ \text{thm}}}{=} \oiint_S \vec{F}\cdot\hat{n}\, dS = \text{amount of smoke through } S \text{ per unit time}$$

$$= -\frac{d}{dt}\Big(\underbrace{\iiint_D u\, dV}_{\text{amount of smoke contained in } D}\Big).$$

$$= -\iiint_D \frac{\partial u}{\partial t}\, dV.$$

Hence, for any D, $\iiint_D \operatorname{div}\vec{F}\, dV = -\iiint_D \frac{\partial u}{\partial t}\, dV$.

$$\Rightarrow \boxed{\operatorname{div}\vec{F} = -\frac{\partial u}{\partial t}}$$

- Therefore, $\frac{\partial u}{\partial t} = -\operatorname{div}\vec{F} = -\operatorname{div}(-k\nabla u) = k\nabla^2 u$. □

3. Archimede's principle

Suppose a body is fully or partially submerged in a fluid, there is a buoyant force $\vec{F}$ acts on the body.

Then, $\vec{F}$ is upwards; $|\vec{F}|$ = weight of the fluid displaced by the body.