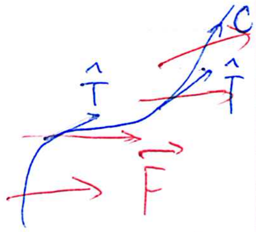
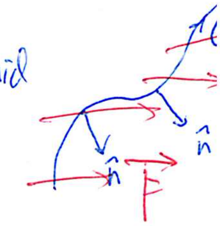
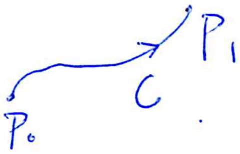
Line Integrals

	Scalar function	Vector field (tangential)	Vector field (normal)
	$\int_C f\,ds$	$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{T}\,ds$	$\int_C \vec{F}\cdot\hat{n}\,ds$
Interpretation	mass of wire C with density $\delta = f$	work of $\vec{F}$ along C. 	**Flux**: how much fluid passes through C 
Computation via parametrization $C: \vec{r}(t)$ $a\le t\le b$	$\int_C f\,ds = \int_a^b f(\vec{r}(t))\left\lvert\frac{d\vec{r}}{dt}\right\rvert dt$	$\vec{F} = \langle M, N\rangle$ $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(t)\cdot\frac{d\vec{r}}{dt}\,dt$ $= \int_C M\,dx + N\,dy$	$\vec{F} = \langle M, N\rangle$ $\int_C \vec{F}\cdot\hat{n}\,ds = \int_C -N\,dx + M\,dy$
Fundamental theorem of Calculus for line integrals	N.A.	If $\vec{F} = \nabla f$ gradient field $\int_C \vec{F}\cdot d\vec{r} = f(P_1) - f(P_0)$. 	N.A
Green's theorem (relation to double integral)	N.A.	 closed curve counter-clockwise. $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA$ or. $\oint_C M\,dx + N\,dy = \iint_R \underbrace{(N_x - M_y)}_{= \text{curl}\,\vec{F} \text{ for } \vec{F}=\langle M,N\rangle}\,dA$	$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \text{div}\,\vec{F}\,dA$ or. $\oint_C -N\,dx + M\,dy = \iint_R \underbrace{(M_x + N_y)}_{= \text{div}\,\vec{F} \text{ for } \vec{F}=\langle M,N\rangle}\,dA$

Equivalent Properties:

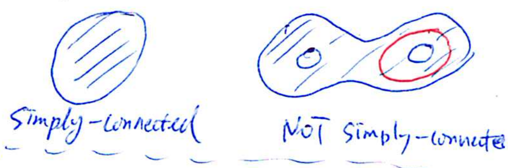
Let $D \subseteq \mathbb{R}^2$ be a simply connected region

$\vec{F}$ defined everywhere in D.

Then the followings are equivalent.

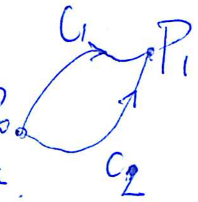
Def: Simply-connected region:

Interior of any closed curve in D is also contained in D



(1) $\vec{F}$ is conservative in D i.e., $\oint_C \vec{F}\cdot d\vec{r} = 0 \quad \forall C \subset D$.

(2) $\vec{F}$ is path-independence in D. i.e., $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} \quad \forall C_1, C_2 \subset D$ with same end pts.



(3) $\vec{F} = \nabla f$ in D.

(4) $\mathrm{curl}\,\vec{F} = N_x - M_y = 0$ in D.

Pf. (1) $\Leftrightarrow$ (2) trivial.

(3) $\Rightarrow$ (2). Fundamental theorem of Calculus for line integrals:

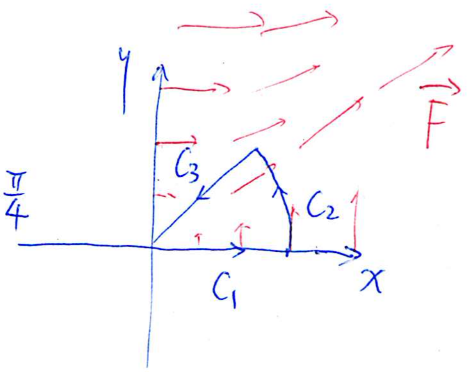
$$\int \nabla f \cdot d\vec{r} = f(P_1) - f(P_0).$$

(2) $\Rightarrow$ (3). How we find potential using line integrals.

(3) $\Rightarrow$ (4). $\mathrm{curl}\,\vec{F} = f_{yx} - f_{xy} = 0$. by mixed derivative theorem

(4) $\Rightarrow$ (1) Green's theorem. $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \mathrm{curl}\,\vec{F}\, dA = 0$.

Example 1: $\vec{F} = \langle y, x \rangle$.



$C = C_1 + C_2 + C_3$ enclosing sector of unit disk, $0 \le \theta \le \frac{\pi}{4}$

Compute $\oint_C \vec{F} \cdot d\vec{r} = \oint_C y\,dx + x\,dy$.

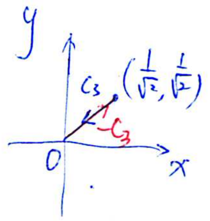
- On C_1: $y = 0$, $dy = 0$. Hence $\int_{C_1} y\,dx + x\,dy = \boxed{0}$.

  Geometrically, $\vec{F} \perp \hat{i}$ on x-axis $\Rightarrow \vec{F} \cdot \hat{T} = 0 \Rightarrow \int_{C_1} \vec{F} \cdot \hat{T}\,ds = 0$.

- On C_2: parametrize $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le \frac{\pi}{4}$.

  then $\int_{C_2} y\,dx + x\,dy = \int_0^{\frac{\pi}{4}} \sin\theta(-\sin\theta)d\theta + \cos\theta(\cos\theta)d\theta = \int_0^{\frac{\pi}{4}} (\cos^2\theta - \sin^2\theta)d\theta$

  $= \boxed{\frac{1}{2}}$



- On C_3: parametrize $x = t$, $y = t$, $0 \le t \le \frac{1}{\sqrt{2}}$ for $-C_3$

  then $\int_{C_3} y\,dx + x\,dy = -\int_{-C_3} y\,dx + x\,dy = -\int_0^{\frac{1}{\sqrt{2}}} 2t\,dt = \boxed{-\frac{1}{2}}$

Total work $= \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + \frac{1}{2} - \frac{1}{2} = \boxed{0}$.

Alternatively, $\operatorname{curl} \vec{F} = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = 0$. $\vec{F}$ is a gradient field.

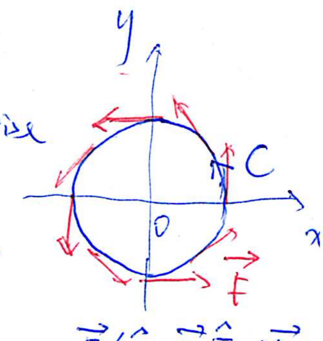
Direct observation $\Rightarrow \vec{F} = \nabla f$ when $f = xy$.

Then $\int_{C_2} \vec{F} \cdot d\vec{r} = f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) - f(1, 0) = \frac{1}{2}$. etc.,

$\oint_C \vec{F} \cdot d\vec{r} = 0$ as C is a closed curve.

□

Example 2: $\vec{F} = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$



$\vec{F} \parallel \hat{T}$, $\vec{F} \cdot \hat{T} = \|\vec{F}\|$

C: circle of radius a centered at origin, counterclockwise

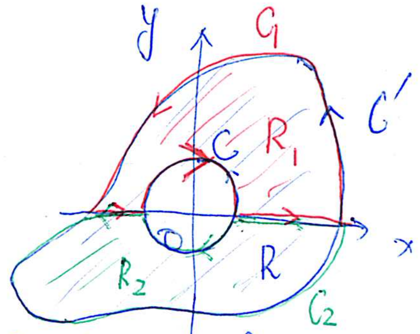
Note $|\vec{F}| = \frac{1}{\sqrt{x^2+y^2}} \to \infty$ near $(0,0)$. In fact, $\vec{F}$ not defined at 0.

$\operatorname{curl} \vec{F} = \frac{\partial\left(\frac{x}{x^2+y^2}\right)}{\partial x} - \left(\frac{\partial\left(-\frac{y}{x^2+y^2}\right)}{\partial y}\right) = 0$

But $\oint_C \vec{F} \cdot d\vec{r} = \oint \vec{F} \cdot \hat{T}\,ds = \oint \frac{1}{a}\,ds = \frac{1}{a} \cdot 2\pi a = \boxed{2\pi} \ne 0$

Example 3: $\vec{F} = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$

C' = arbitrary simple closed curve winding the origin once.



Since $\vec{F}$ not defined at O, cannot apply Green's thm directly.

Instead, take a circle of radius a that contains inside C', and let R be the region bounded by C and C'. Subdivide $R = R_1 \cup R_2$.

Apply Green's theorem on R_i $\Rightarrow$ $\int_{C_i} \vec{F} \cdot d\vec{r} = \iint_{R_i} \operatorname{curl} \vec{F} \, dA$ ($= 0$ in this case)

Clearly. $\iint_R = \iint_{R_1} + \iint_{R_2}$

$\int_{C'} - \int_C = \int_{C_1} + \int_{C_2}$ since cancellation.

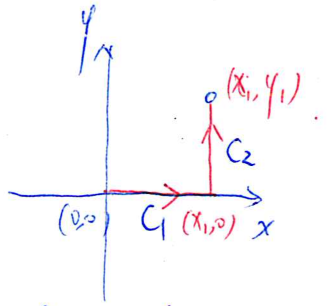
$$\Rightarrow \boxed{\int_{C'} - \int_C = \iint_R}$$

In this example $\Rightarrow$ $\int_{C'} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \iint_R \operatorname{curl} \vec{F} \, dA = 2\pi$ □

Example 4: Finding potential of $\vec{F} = \langle 4x^2 + 8xy, 3y^2 + 4x^2 \rangle$.

(Note that $\frac{\partial(3y^2+4x^2)}{\partial x} - \frac{\partial(4x^2+8xy)}{\partial y} = 0$).

Two methods.



1) Computing line integrals: $f(x_1, y_1) = f(0,0) + \int_C \vec{F} \cdot d\vec{r}$

On C_1. $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{x_1} 4x^2 \, dx = \frac{4}{3} x_1^3$

On C_2. $\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{y_1} (3y^2 + 4x^2) \, dy = y_1^3 + 4x_1^2 y_1$

$\Rightarrow f(x_1, y_1) = \frac{4}{3} x_1^3 + y_1^3 + 4x_1^2 y_1 + C$ — constant $= f(0,0)$.

2) Antiderivatives. Want to solve $\begin{cases} f_x = 4x^2 + 8xy & (1) \\ f_y = 3y^2 + 4x^2 & (2). \end{cases}$

antiderivative w.r.t. x

$(1) \Rightarrow f = \frac{4}{3} x^3 + 4x^2 y + g(y)$, ← integration "constant": indep of x.

Hence. $f_y = 4x^2 + g'(y)$. Matching with (2) $\Rightarrow g'(y) = 3y^2 \Rightarrow g(y) = y^3 + C$ (constant).

Plug back, $f = \frac{4}{3} x^3 + 4x^2 y + y^3 + C$ □