Step 2 : φ is a covering map

We need to show that $\forall\, y \in N$, $\exists$ nbd. U of y in N such that $\varphi^{-1}(U) = \bigcup_i W_i$ with

$$\begin{cases} \cdot\ W_i \cap W_j = \emptyset \text{ for } i \neq j \\ \cdot\ \varphi : W_i \to U \text{ is a diffeomorphism} \end{cases}$$

Pf of Step 2

$\forall\, y \in N$, $\exists\, \delta > 0$ such that

$$\exp_y^N : B^N(\delta) \to B_\delta^N \quad \text{is a diffeomorphism}$$

where
$$B^N(\delta) = \{ v \in T_y N : |v|_N < \delta \}$$
$$B_\delta^N = \{ z \in N : d_N(z, y) < \delta \}.$$

Since φ is a local isom. & hence a local diffeo, $\varphi^{-1}(y)$ is a discrete set in M. Let $\varphi^{-1}(y) = \{x_i\}_{i \in \Lambda}$ for some index set Λ, and denote

$$B^i(\delta) = B^M(x_i, \delta) = \{ v \in T_{x_i} M : |v|_M < \delta \}$$
$$B^i_\delta = B^N_\delta(x_i) = \{ \xi \in M : d_M(\xi, x_i) < \delta \}.$$

Claim : (i) $\varphi^{-1}(B_\delta^N) = \bigcup_i B_\delta^i$

(ii) $\forall\, i$, $\varphi : B_\delta^i \to B_\delta^N$ is a diffeo.

(iii) $\forall\, i \neq j$, $B_\delta^i \cap B_\delta^j = \emptyset$.

<u>Pf of (i)</u>: It is clear that $\bigcup_i B_\delta^i \subset \varphi^{-1}(B_\delta^N)$ since φ is a local isom. Conversely, for $z \in \varphi^{-1}(B_\delta^N)$, we have $\varphi(z) \in B_\delta^N$. By the choice of $\delta > 0$, $\exists$ unique geodesic $\gamma : [0,1] \to B_\delta^N$ such that

$$\gamma(0) = \varphi(z) \quad \& \quad \gamma(1) = y.$$

Then by the argument in the proof of Step 1, $\exists$ a geodesic $\tilde{\gamma} : [0,1] \to M$ such that

$$\tilde{\gamma}(0) = z \quad \& \quad \varphi(\tilde{\gamma}(t)) = \gamma(t),\ \forall\, t.$$

$\Rightarrow \quad \varphi(\tilde{\gamma}(1)) = \gamma(1) = y$

$\Rightarrow \quad \tilde{\gamma}(1) \in \varphi^{-1}(y) = \{x_i\}_{i \in \Lambda}$.

$\Rightarrow \quad \tilde{\gamma}(1) = x_i$ for some $i \in \Lambda$.

Again, using φ = local isom., we have

$$\mathrm{Length}_M(\tilde{\gamma}) = \mathrm{Length}_N(\gamma) < \delta$$

$\Rightarrow \quad \tilde{\gamma}(0) = z$ has a distance $< \delta$ to x_i

$$\Rightarrow \qquad z \in B^i_\delta \subset \bigcup_i B^i_\delta .$$

This proves (i).

<u>Pf of (ii)</u> : By the note in Step 1, we have

$$\begin{tikzcd} B^i(\delta) \arrow[r, "d\varphi"] \arrow[d, "\exp^M_{x_i}"'] \arrow[dr, phantom, "\circlearrowright"] & B^N(\delta) \arrow[d, "\exp^N_y"] \\ B^i_\delta \arrow[r, "\varphi"] & B^N_\delta \end{tikzcd}$$

(since φ = local isom.)

i.e. $\qquad \varphi \circ \exp^M_{x_i} = \exp^N_y \circ d\varphi .$

By the choice of $\delta > 0$, $\exp^N_y$ and $d\varphi$ are diffeomaphisms. Hence $\exp^M_{x_i}$ has to be an immersion. On the other hand $\exp^M_{x_i} : B^i(\delta) \to B^i_\delta$ is surjective (since M is complete), therefore we have

$$\varphi = \exp^N_y \circ d\varphi \circ \left(\exp^M_{x_i}\right)^{-1}$$

which is a diffeomaphism. This proves (ii).

<u>Pf of (iii)</u> : Let $i \neq j \in \Lambda$. Suppose that $B^i_\delta \cap B^j_\delta \neq \emptyset$.

Then $\exists\ \xi \in B^i_\delta \cap B^j_\delta$. Using (ii), $\exists$ geodesics

$\tilde{\gamma}_i \in B_\delta^i$ & $\tilde{\gamma}_j \in B_\delta^j$

joining ξ to x_i & x_j respectively.

Then $\varphi(\tilde{\gamma}_i)$ & $\varphi(\tilde{\gamma}_j)$ are geodesics in B_δ^N

joining $\varphi(\xi)$ and $\varphi(x_i) = y = \varphi(x_j)$.

$\Rightarrow$ $\varphi(\tilde{\gamma}_i) = \varphi(\tilde{\gamma}_j) = \gamma$ the unique geodesic in B_δ^N

joining $\varphi(\xi)$ to y.

Therefore $\tilde{\gamma}_i, \tilde{\gamma}_j$ are both liftings of γ passing thro

a common point ξ, we have $\tilde{\gamma}_i = \tilde{\gamma}_j$.

$\Rightarrow$ $x_i = \tilde{\gamma}_i(1) = \tilde{\gamma}_j(1) = x_j$ contradiction.

This proves (iii).

By this claim, B_δ^N is the required (uniform) nbd.

of y. $\therefore$ φ is a covering map. ※

<u>Lemma 9</u>: Let
- M = complete Riem. mfd.
- $x \in M$ s.t.
- $\exp_x : T_xM \to M$ has no conjugate point.

Then $\exp_x$ is a covering map.

Pf: Let g = Riem. metric on M.

Denote $\tilde{g} = (\exp_x)^* g$ be the pull-back metric of g by $\exp_x$ on T_xM. (since $\exp_x$ has no conjugate point.)

i.e. $\tilde{g}(X, Y) \overset{\text{def}}{=} g((d\exp_x)(X), (d\exp_x)(Y))$

$\forall X, Y \in \Gamma(T_xM)$.

Claim: $\tilde{g}$ is a complete metric on T_xM.

Pf of claim: Note that Euclidean rays (from 0) in T_xM can be parametrized by

$$\begin{aligned} \tilde{\gamma} : [0,\infty) &\longrightarrow T_xM \\ t &\longmapsto tv \end{aligned} \qquad (\text{for some } v \in T_xM)$$

By def. of $\exp_x$, $\exp_x(\tilde{\gamma}(t))$ is a geodesic in M starting at x. Therefore, by definition of $\tilde{g} = (\exp_x)^* g$, $\tilde{\gamma}(t)$ is a geodesic of $\tilde{g}$ starting from 0. This implies geodesic from $0 \in T_xM$ is defined $\forall t \in [0,\infty)$. Hence

$$\exp_0^{T_xM} : T_0(T_xM) \to (T_xM, \tilde{g})$$

is defined on the whole $T_0(T_xM)$. Therefore Hopf-Rinow Thm $\Rightarrow (T_xM, \tilde{g})$ is complete.

This proves the claim.

Now by the claim and the assumption that $\exp_x$ has no conjugate point, $\exp_x : (T_xM, \tilde{g}) \to (M, g)$ is a local isometry from a complete Riem. mfd.

Therefore, Lemma 8 $\Rightarrow$ $\exp_x : T_xM \to M$ is a covering.

$\#$

Pf of (2) of Cartan-Hadamard:

By Lemma 9, $\exp_x : T_xM \to M$ is a covering. Together with the assumption that M is simply-connected, we have proved that $\exp_x$ is a diffeomorphism. $\#$

Thm 10 Let M, N = simply-connected n-dim'l space forms with constant sectional curvature K. Let $x \in M$ and $y \in N$ and $\{e_1, \cdots, e_n\} \subset T_xM$ and $\{\varepsilon_1, \cdots, \varepsilon_n\} \subset T_yN$ are orthonormal bases respectively. Then $\exists$ unique isometry $\varphi : M \to N$ such that

$$\begin{cases} \varphi(x) = y & \text{and} \\ d\varphi(e_i) = \varepsilon_i, & \forall i \end{cases}.$$

Note: Thm 10 $\Rightarrow$ uniqueness of the Thm 1 in Ch5.

We need the following lemma 11 & 12:

Lemma 11: Let
- M = n-dim'l space form
- constant sectional curvature K
- $x \in M$, $\{e_1, \cdots, e_n\} \subset T_xM$ ortho. basis.

Then the curvature tensor satisfies

$$R_{e_i e_j} e_k = K(\delta_{ik} e_j - \delta_{jk} e_i), \quad \forall i, j, k = 1, \cdots, n.$$

Pf: Define $\tilde{R}$ by the RHS i.e

$$\tilde{R}_{e_i e_j} e_k \overset{\text{def}}{=\!=} K(\delta_{ik} e_j - \delta_{jk} e_i)$$

Then $\tilde{R}$ can be extended to a tensor (Ex!) satisfying all the symmetric properties of the curvature tensor (i.e. (1) — (4) in Lemma 1 of §3.3) (Ex!)

Furthermore, for tangent vectors v & w with $|v| = |w| = 1$ and $\langle v, w \rangle = 0$, one has $\langle \tilde{R}_{vw} v, w \rangle = K$ (Ex!)

Therefore Lemma 2 of §3.3 $\Rightarrow$ $\tilde{R} \equiv R$. ✗✗

<u>Lemma 12</u>: Same assumption as in Lemma 11

Let • $v \in T_x M$ with $|v| = 1$

• $v^\perp$ = orthogonal complement of v

Then
$$R_{vw} v = \begin{cases} Kw, & \text{if } w \in v^\perp \\ 0, & \text{if } w = cv, \text{ for some } c \in \mathbb{R}. \end{cases}$$

(Pf: Straight forward from Lemma 11.)

<u>Pf of Thm 10</u>: It is clear that we only need to show the cases of $K = 0, +1$ or -1. And we may assume $M = \mathbb{R}^n$, S^n or $\mathbb{H}^n$.

<u>Case 1</u>: <u>$K = 0$ or -1</u>

Since $K \le 0$, Cartan-Hadamard $\Rightarrow$

$$\begin{cases} \exp_x^M : T_x M \to M \\ \exp_y^N : T_y N \to N \end{cases} \text{ are diffeomorphisms.}$$

Let $\Phi: T_xM \to T_yN$ be the unique isometry between the inner product spaces T_xM & T_yN such that

$$\Phi(e_i) = \varepsilon_i \quad , \forall\, i = 1, \cdots, n .$$

Define $\varphi: M \to N$ by

$$\varphi = \exp_y^N \circ \Phi \circ (\exp_x^M)^{-1}$$

$$\begin{tikzcd} T_xM \arrow[r, "\Phi"] \arrow[d, "\exp_x^M"'] & T_yN \arrow[d, "\exp_y^N"] \\ M \arrow[r, dashed, "\varphi"] & N \end{tikzcd}$$

Clearly φ is a diffeomorphism. We need to show that φ is an isometry. i.e. $\forall\, z \in M$ and $X \in T_zM$, we have

$$|d\varphi(X)|_N = |X|_M .$$

By Cartan-Hadamard,

$\exists\, T \in T_xM$ and $w \in T_T(T_xM) \cong T_xM$ s.t.

$$z = \exp_x^M(T) \quad \text{and} \quad X = (d\exp_x^M)_T(w)$$

Then we can define a 1-parameter family of geodesics

$$\gamma_u(t) = \exp_x^M [t(T + uw)] .$$

Let $V(t)$ = transversal vector field of γ_u along γ_0.

Then $V(t)$ is a Jacobi field s.t.

$$\begin{cases} V(0) = 0 \\ V'(0) = w \end{cases}$$

and further $V(1) = (d\exp_x^M)_T(w) = X$.

In N, we define correspondingly

$$\gamma_u^N(t) = \exp_y^N\left[t\left(\Phi(T) + u\,\Phi(w)\right)\right]$$

& $V^N(t)$ = transversal vector field of $\{\gamma_u^N\}$ along γ_0^N.

Then V^N is a Jacobi field along $\gamma_0^N \subset N$ s.t.

$$\begin{cases} V^N(0) = 0 \\ (V^N)'(0) = \Phi(w). \end{cases}$$

Note that

$$\begin{aligned} \varphi(\gamma_u(t)) &= \left[\exp_y^N \circ \Phi \circ (\exp_x^M)^{-1}\right]\left(\exp_x^M[t(T+uw)]\right) \\ &= \exp_y^N \circ \Phi\left(t(T+uw)\right) \\ &= \exp_y^N\left[t(\Phi(T) + u\,\Phi(w))\right] = \gamma_u^N(t) \end{aligned}$$

$$\Rightarrow \quad d\varphi(\mathcal{V}(t)) = \mathcal{V}^N(t) \qquad \text{(by differentiation)}$$

$$\Rightarrow \quad \mathcal{V}^N(1) = d\varphi(\mathcal{V}(1)) = d\varphi(X).$$

Therefore, we need to show that

$$|\mathcal{V}^N(1)|_N = |\mathcal{V}(1)|_M.$$

To see this, we use parallel orthonormal frames $\{e_1(t), \cdots, e_n(t)\}$ & $\{\varepsilon_1(t), \cdots, \varepsilon_n(t)\}$ along γ_0 and γ_0^N respectively such that

$$\begin{cases} e_i(0) = e_i \\ \varepsilon_i(0) = \varepsilon_i \end{cases} \quad \forall\, i = 1, \cdots, n.$$

Then
$$\begin{cases} \mathcal{V}(t) = \sum_i f_i(t)\, e_i(t) \\ \mathcal{V}^N(t) = \sum_i g_i(t)\, \varepsilon_i(t) \end{cases} \qquad \text{for some functions } f_i(t), g_i(t).$$

Furthermore, $\mathcal{V}(0) = 0$ & $\mathcal{V}'(0) = w \Rightarrow$

$$\begin{cases} f_i(0) = 0 \\ f_i'(0) = \langle w, e_i \rangle \end{cases}$$

(by lemma 12) (Ex!)

$$\therefore \quad f_i'' + \sum_j f_j K \left[|T|^2 \delta_{ij} - \langle T, e_i \rangle \langle T, e_j \rangle \right] = 0.$$

$$(*) \begin{cases} f_i(0)=0 \\ f_i'(0)=\langle w, e_i\rangle \end{cases}$$

Similarly, we have

$$\begin{cases} g_i'' + \sum_j g_j K \left[|\Phi(T)|^2 \delta_{ij} - \langle \Phi(T), \varepsilon_i\rangle \langle \Phi(T), \varepsilon_j\rangle \right] = 0 \\ g_i(0)=0 \\ g_i'(0) = \langle \Phi(w), \varepsilon_i\rangle \end{cases}$$

Using the fact Φ is an isometry (between inner product spaces T_xM & T_yN) we have

$$\begin{cases} |\Phi(T)|^2 = |T|^2 \\ \langle \Phi(T), \varepsilon_i\rangle = \langle \Phi(T), \Phi(e_i)\rangle = \langle T, e_i\rangle \\ \langle \Phi(w), \varepsilon_i\rangle = \langle w, e_i\rangle \end{cases}$$

$\therefore$ $\{f_i\}$ & $\{g_i\}$ satisfy the same IVP of an ODE system $(*)$, therefore $f_i \equiv g_i$, $\forall t$, $i=1,\cdots,n$.

Hence $|\tilde{V}^N(1)|^2 = \sum_i g_i^2(1) = \sum_i f_i^2(1) = |\tilde{V}(1)|^2$.

This proves the case that $K=0$ or -1.

Case of $K=+1$

We may assume $M=\mathbb{S}^n$

If $\bar{x}=-x$ (the antipodal point of x), then

$(\exp_x^M)^{-1}: \mathbb{S}^n\setminus\{\bar{x}\} \to T_x\mathbb{S}^n$ is well-defined.

Therefore, we can define similarly the map

$$\varphi = \exp_y^N \circ \Phi \circ (\exp_x^M)^{-1} : \mathbb{S}^n\setminus\{\bar{x}\} \to N.$$

Similar argument shows that φ is a <u>local</u> isometry

Observes that $\forall\, z\in \mathbb{S}^n\setminus\{x,\bar{x}\}$, we still have

$$\begin{tikzcd}
T_z\mathbb{S}^n \arrow[r, "d\varphi"] & T_{\varphi(z)}N \arrow[d, "\exp^N_{\varphi(z)}"] \\
\mathbb{S}^n\setminus\{\bar{x},\bar{z}\} \arrow[u, "(\exp_z^M)^{-1}"] \arrow[r, "\varphi"] \arrow[ur, phantom, "\circlearrowleft"] & N
\end{tikzcd}$$

(as φ is a local isom.)

Note that $d\varphi\big|_{T_z\mathbb{S}^n}: T_z\mathbb{S}^n \to T_{\varphi(z)}N$ is an inner product space isometry, same argument above implies that $\psi : \mathbb{S}^n\setminus\{\bar{z}\}\to N$ defined by

$$\psi \stackrel{\text{def}}{=\!=} \exp^N_{\varphi(z)} \circ \left(d\varphi\big|_{T_z S^n}\right) \circ \left(\exp^{S^n}_z\right)^{-1}$$

is a local isometry. By the above commutative diagram, $\forall\ p \in S^n \setminus \{\bar{x}, \bar{z}\}$

$$\begin{aligned}\varphi(p) &= \exp^N_{\varphi(z)} \circ d\varphi \circ \left(\exp^{S^n}_z\right)^{-1}(p) \\ &= \exp^N_{\varphi(z)} \circ \left(d\varphi\big|_{T_z S^n}\right) \circ \left(\exp^{S^n}_z\right)^{-1}(p) \\ &= \psi(p).\end{aligned}$$

Therefore, we can extend φ to be defined on the whole S^n by setting $\varphi(\bar{x}) = \psi(\bar{x})$.

Then by the construction of $\varphi : S^n \to N$ is a local isometry. Similar argument as in lemma 8 $\Rightarrow$ φ is a covering map. Since N is simple-connected, φ has to be an isometry.

Finally, it is clear that $d\varphi(e_i) = \varepsilon_i$, $\forall\ i = 1, \cdots, n$.

So we've proved the existence part of Thm 10.

For uniqueness: we first prove

<u>Lemma 13</u>: Let $\varphi_i : M \to N$, $i=1,2$, be 2 local isometries between complete Riem. mfds. M & N such that for some $x \in M$,

$$\begin{cases} \varphi_1(x) = \varphi_2(x) \\ d\varphi_1\big|_{T_xM} = d\varphi_2\big|_{T_xM} \end{cases}$$

Then $\varphi_1 \equiv \varphi_2$.

<u>Pf</u>: Let $S = \{ z \in M : \varphi_1(z) = \varphi_2(z)$ & $d\varphi_1|_{T_zM} = d\varphi|_{T_zM} \}$

- By assumption, $x \in S$. $\therefore S \neq \emptyset$.
- It is clear that S is closed by continuity.
- If $z \in S$, take $\delta > 0$ s.t.

$$\exp_z^M : B(\delta) \to M \quad \text{is a diffeo. injection.}$$

Recall that we have

$$\begin{tikzcd} T_zM \arrow[r, "d\varphi"] \arrow[d, "\exp_z^M"'] \arrow[dr, phantom, "\mathcal{R}"] & T_{\varphi(z)}N \arrow[d, "\exp_{\varphi(z)}^N"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

$\forall$ local isometry φ.

Applying this to φ_1 & φ_2, we have

$$\exp_z^M(B(\delta)) \subset S' \qquad (\text{Ex!})$$

$\Rightarrow$ S' is open.

Therefore, by <u>connectedness</u> of $M \Rightarrow S' = M$. ⋕

<u>Pf of Uniqueness of Thm 10</u> : Immediately from Lemma 13. ⋕

<u>Cor 14</u> : Let M = complete simply-connected Riem. mfd. of dim $= n$.

Then M is a space form

$\Leftrightarrow$ $\forall\, x, y \in M$ and

$\forall$ orthonormal bases $\{e_i\}$ of T_xM & $\{\varepsilon_i\}$ of T_yM,

$\exists$ isometry $\varphi : M \to M$ s.t. $\varphi(x) = y$ and $d\varphi(e_i) = \varepsilon_i$, $\forall i$

(Pf: Immediately from Thm 10)

<u>Note</u>: Cor 14 proves that simply-connected space form is <u>homogeneous</u>. In fact, we have more

Cor 15 <u>Simply-connected space forms are two-points homogeneous.</u>

Def: M is called two-points homogeneous if $\forall\, p_1, p_2, q_1, q_2 \in M$ with $d(p_1,p_2) = d(q_1,q_2)$ $\exists$ an isometry $\varphi : M \to M$ such that

$$\varphi(p_1) = q_1 \ \& \ \varphi(p_2) = q_2 .$$

Pf of Cor 15 Let p_1, p_2, q_1, q_2 be points in a simply-connected space form M s.t.

$$d(p_1,p_2) = d(q_1,q_2) = \alpha .$$

Let $\zeta, \xi : [0,\alpha] \to M$ be normalized geodesics s.t.

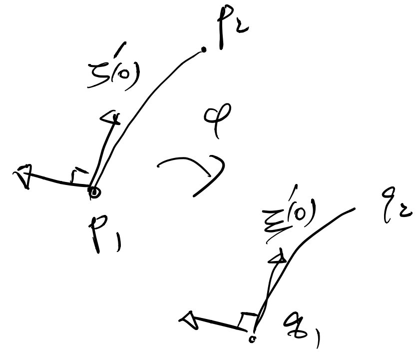
$$\zeta(0) = p_1,\ \zeta(\alpha) = p_2$$

$$\xi(0) = q_1,\ \xi(\alpha) = q_2$$

Choose orthonormal bases

$\{e_i\}$ on $T_{p_1}M$ s.t. $e_1 = \zeta'(0)$ &

$\{\varepsilon_i\}$ on $T_{q_1}M$ s.t. $\varepsilon_1 = \xi'(0)$.



Then Thm 10 (or Cor 14) $\Rightarrow \exists$ isometry $\varphi: M \to M$ s.t. $\varphi(p_1) = q_1$ and $d\varphi(e_i) = \varepsilon_i$

$\Rightarrow$ $\varphi \circ \zeta$ & ξ are geodesics with the same initial data, hence $\varphi \circ \zeta = \xi$.

$\Rightarrow$ $\varphi(p_2) = q_2$ ※

Pf of (*) in the proof of Thm 10 :

We need to calculate the curvature term

$$R_{\gamma_0'(t) U(t)} \gamma_0'(t).$$

Let $V_0(t) = \dfrac{\gamma_0'(t)}{|\gamma_0'(t)|}$, then

$$R_{\gamma_0'(t) U(t)} \gamma_0'(t) = |\gamma_0'(t)|^2 R_{V_0(t) U(t)} V_0(t)$$

$$(\text{Lemma 12}) = |\gamma_0'(t)|^2 K \left[U(t) - \langle U(t), V_0(t) \rangle V_0(t) \right]$$

Since $\langle \gamma_0'(t), \gamma_0'(t) \rangle = \langle \gamma_0'(0), \gamma_0'(0) \rangle = |T|^2$,

$\langle \gamma_0'(t), e_i(t) \rangle = \langle T, e_i \rangle$,

we have

$$V''(t) + R_{\gamma_0'(t) V(t)} \gamma_0'(t) = 0$$

$$\Leftrightarrow \quad \sum f_i'' e_i + |\gamma_0'|^2 K \left[\sum f_i e_i - \frac{\langle \sum f_i e_i, \gamma_0' \rangle}{|\gamma_0'|^2} \gamma_0' \right] = 0$$

$$\Leftrightarrow \quad \sum_i \left(f_i'' + |T|^2 K f_i \right) e_i - K \sum_i f_i \langle e_i, \gamma_0' \rangle \gamma_0' = 0$$

$$\Leftrightarrow \quad \sum_i \left(f_i'' + |T|^2 K f_i \right) e_i - K \sum_i f_i \langle e_i, T \rangle \sum_j \langle e_j, \gamma_0' \rangle e_j = 0$$

$$\Leftrightarrow \quad \sum_i \left(f_i'' + |T|^2 K f_i \right) e_i - K \sum_{i,j} f_i \langle e_i, T \rangle \langle e_j, T \rangle e_j = 0$$

$$\Leftrightarrow \quad \sum_i \left[f_i'' + |T|^2 K f_i - K \sum_{i,j} f_j \langle e_j, T \rangle \langle e_i, T \rangle \right] e_i = 0$$

$$\Leftrightarrow \quad f_i'' + \sum_j f_j K \left[|T|^2 \delta_{ij} - \langle e_j, T \rangle \langle e_i, T \rangle \right] = 0$$

$\forall i = 1, \cdots, n.$

#