Ch5 Isometry, Space Forms

(M,g) = complete Riemannian manifold (connected)

Def: (M,g) with constant sectional curvature is called a space form.

Thm 1: $\forall\, c \in \mathbb{R}$ & $n \geq 2$, $\exists$ unique (up to isometry) simply-connected space form of dimension n and with constant sectional curvature c.

egs (Proof later)

- $c = 0$, $(\mathbb{R}^n$, standard flat metric$)$
- $c = +1$, $(S^n$, standard metric$)$
- $c = -1$, $\left(B^n, \dfrac{4}{\left[1-\sum_{i=1}^{n}(x^i)^2\right]^2} dx^1\otimes dx^1 + \cdots + dx^n \otimes dx^n\right)$

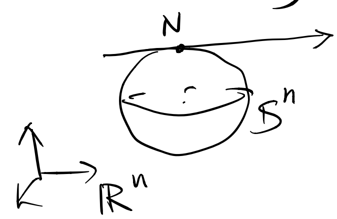
  where $B^n = \{(x^1, \cdots, x^n) : \sum_{i=1}^{n}(x^i)^2 < 1\}$

  (Hyperbolic n-space : unit ball model)

Def: Let M be a submanifold of $\overline{M}$ equipped with the induced metric. Then M is called a totally geodesic submanifold of $\bar{M}$ if a geodesic γ (of $\bar{M}$) tangents to M implies $\gamma \subset M$.

Note: Such a geodesic γ of $\bar{M}$ must be a geodesic of the submanifold M.

egs:
- $\mathbb{R}^k \hookrightarrow \mathbb{R}^n : (x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$ gives a totally geodesic submanifold of $\mathbb{R}^n$.
- $S^n \subset \mathbb{R}^{n+1}$ is not a totally geodesic submanifold as tangent lines to S^n don't stay on S^n.



Let
- $M \subset \bar{M}$ be a submanifold
- M equipped with induced metric
- $D, \bar{D}$ = Levi-Civita connections of $M, \bar{M}$ respectively

$$\left(D_X Y = (\bar{D}_X Y)^{\text{tangential part}} \quad \forall X, Y \in \Gamma(TM) \subset \Gamma(T\bar{M}) \right)$$

Consider
$$S(X,Y) = D_X Y - \bar{D}_X Y, \quad \forall X, Y \in \Gamma(TM)$$

(Note: S defines for vector fields on M, not $\bar{M}$)

Facts : • $S(X_1+X_2, Y) = S(X_1, Y) + S(X_2, Y)$

• $S(X,Y) = S(Y,X)$

• $\forall f \in C^\infty(M)$, $S(fX,Y) = S(X,fY) = fS(X,Y)$

$\therefore$ S is "symmetric" tensor on M.

Pf of symmetry : $S(X,Y) - S(Y,X)$

$$= (D_X Y - \bar{D}_X Y) - (D_Y X - \bar{D}_Y X)$$

$$= (D_X Y - D_Y X) - (\bar{D}_X Y - \bar{D}_Y X)$$

$$= [X,Y] - [X,Y] \quad (D, \bar{D} : \text{Levi-Civita})$$

$$= 0 . \quad ※$$

Therefore, we can define a symmetric bilinear form on T_xM, $\forall x \in M$:

$\forall v, w \in T_xM$, $S_x(v,w) = S(V,W)(x)$

where V, W = any extension of v, w.

Def : This S is called the <u>2^{nd} fundamental form</u> of M in $\bar{M}$.

<u>Lemma 2</u> $M \subset \overline{M}$ totally geodesic

$\Longleftrightarrow$ $S \equiv 0$, where $S = 2^{nd}$ f.f. of M in $\overline{M}$.

(i.e. $D_X Y = \overline{D}_X Y$, $\forall X, Y \in \Gamma(TM)$)

<u>Pf</u>: ($\Rightarrow$) Let $x \in M$ & $v \in T_x M \subset T_x \overline{M}$

Let γ = geodesic on $\overline{M}$ with

$\gamma(0) = x$, $\gamma'(0) = v$.

$\Rightarrow$ $\overline{D}_{\gamma'} \gamma' = 0$.

By assumption, γ is also a geodesic of M

$\Rightarrow$ $D_{\gamma'} \gamma' = 0$.

Therefore $S(v,v) = S(\gamma'(0), \gamma'(0))$

$= D_{\gamma'}\gamma' - \overline{D}_{\gamma'}\gamma' = 0$.

Symmetry of S $\Rightarrow$ $S(v,w) = 0$ $\forall v, w \in T_x M$.

($\Leftarrow$) Suppose $S \equiv 0$

Let γ = geodesic of $\overline{M}$ such that

$\gamma(0) = x$ and $\gamma'(0) = v \in T_x M \subset T_x \overline{M}$.

By Existence and Uniqueness of geodesic in M,

$\exists\, \xi$ = geodesic of M s.t. $\xi(0) = x$, $\xi'(0) = v \in T_x M$

(and of course $\xi \subset M$).

Then $S \equiv 0$

$$\Rightarrow \quad \overline{D}_{\zeta'(t)}\zeta'(t) = D_{\zeta'(t)}\zeta'(t) = 0$$

$\Rightarrow$ ζ is also a geodesic of $\overline{M}$

Then uniqueness of geodesic (in $\overline{M}$)

$$\Rightarrow \gamma = \zeta \subset M .$$

#

<u>Lemma 3</u> Let $M \subset \overline{M}$ be totally geodesic,

$K, \overline{K}$ = sectional curvatures of $M, \overline{M}$ respectively

Then $\forall x \in M$, $\forall$ 2-plane $\pi \subset T_xM \subset T_x\overline{M}$,

$$K(\pi) = \overline{K}(\pi) .$$

(Pf : Immediately from Lemma 2)

<u>eg</u> : Let $\gamma : (a,b) \to \overline{M}$ be a smooth curve parametrized by arc-length. Suppose $\exists$ isometry $\varphi : \overline{M} \to \overline{M}$ such that $\gamma((a,b)) = \{ y \in \overline{M} : \varphi(y) = y \}$.

Then γ is a <u>normalized geodesic</u>.

<u>Pf</u> : We first note that $\forall$ geodesic ζ in $\overline{M}$,

$\varphi \circ \zeta$ is also a geodesic in $\overline{M}$ (since φ = isom.)

Now $\forall\, t_0 \in (a,b)$, take a geodesic

$\xi \subset \overline{M}$ s.t. $\begin{cases} \xi(0) = \gamma(t_0) \\ \xi'(0) = \gamma'(t_0) \end{cases}$

Since $\gamma((a,b))$ = fixed point set of φ,

$d\varphi(\gamma'(t_0)) = \gamma'(t_0)$ (diff. $\varphi \circ \gamma = \gamma$)

$\Rightarrow \quad d\varphi(\xi'(0)) = \xi'(0)$

$\Rightarrow \quad (\varphi \circ \xi)'(0) = \xi'(0)$ (since $\varphi(\xi(0)) = \xi(0)$)

Uniqueness of geodesic $\Rightarrow \varphi \circ \xi = \xi$

$\Rightarrow \quad \xi \subset \{y \in \overline{M} : \varphi(y) = y\} = \gamma((a,b))$

$\Rightarrow \quad \gamma$ is a normalized geodesic. $\#$

<u>Lemma 4</u>: The set of fixed points of an isometry is a totally geodesic submanifold.

(not necessary connected)

<u>Pf</u>: Let $\varphi : \overline{M} \to \overline{M}$ be an isometry and

$M = \{y \in \overline{M} : \varphi(y) = y\}$ be the set of fixed points of φ.

If M is a submanifold of $\overline{M}$, then the same

(Ex.) argument as in the example implies M is totally geodesic. So we only need to show the following

Claim: Let $x \in M$, $B(\delta) = \{ v \in T_x\overline{M} : |v| < \delta \}$

$$B_\delta = \{ y \in \overline{M} : d(x,y) < \delta \}$$

where $\delta > 0$ small enough s.t.

$$\exp_x : B(\delta) \to B_\delta \quad \text{is a diffeomorphism}$$

$(B_\delta = \exp_x B(\delta) .)$

Let $\mathcal{F} \subset T_x\overline{M}$ be a linear subspace defined by

$$\mathcal{F} = \{ v \in T_x\overline{M} : d\varphi(v) = v \}$$

Then

$$M \cap B_\delta = \exp_x (\mathcal{F} \cap B(\delta))$$

Hence M is submanifold of $\overline{M}$.

Pf of Claim:

(1) $M \cap B_\delta \subset \exp_x (\mathcal{F} \cap B(\delta))$

Pf: Let $y \in M \cap B_\delta \subset B_\delta$

$\Rightarrow \exists\, v \in B(\delta)$ s.t. $\exp_x v = y$.

Let $\gamma(t) = \exp_x(tv) : [0,1] \to \overline{M}$

be the unique minimizing geodesic joining x to y.

Since $x, y \in M$, we have $\varphi(x) = x$ & $\varphi(y) = y$.

$\Rightarrow$ $\varphi \circ \gamma$ is also a minimizing geodesic joining x to y.

Uniqueness $\Rightarrow$ $\varphi \circ \gamma = \gamma$

$\Rightarrow$ $d\varphi(v) = v$

$\Rightarrow$ $v \in \mathcal{F}$

$\therefore$ $y = \exp_x v \in \exp_x(\mathcal{F} \cap B(\delta))$.

(2) $\exp_x(\mathcal{F} \cap B(\delta)) \subset M \cap B_\delta$

Pf: Let $y \in \exp_x(\mathcal{F} \cap B(\delta))$. (Then $y \in B_\delta$ by choose of $\delta > 0$.)

Then $\exists\, v \in \mathcal{F} \cap B(\delta)$ such that

$$y = \exp_x v.$$

Let $\gamma(t) = \exp_x(tv) : [0,1] \to \overline{M}$ be the unique minimizing geodesic joining x to y.

Since $v \in \mathcal{F}$, $d\varphi(\gamma'(0)) = \gamma'(0)$

$\Rightarrow$ $\varphi \circ \gamma$ and γ have the same initial data

Uniqueness $\Rightarrow$ $\varphi \circ \gamma = \gamma$

$\Rightarrow$ $y = \gamma(1) = \varphi \circ \gamma(1) = \varphi(y)$

$\therefore$ $y \in M \cap B_\delta$ #

Lemma 5 : $S^n \subset \mathbb{R}^{n+1}$ has constant sectional curvature $+1$, $\forall n \geq 2$.

Pf : "$n=2$" is proved in undergrad DG (Ex.)

If $n \geq 3$, define

$$\tilde{\varphi} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$
$$(x^1, x^2, x^3, x^4, \dots x^{n+1}) \mapsto (x^1, x^2, x^3, -x^4, \dots, -x^{n+1})$$

Then $|\tilde{\varphi}(x)| = |x|$ (Euclidean norm)

Hence $\tilde{\varphi}$ induces an isometry

$$\varphi : S^n \to S^n .$$

The fixed points set

$$M = \{ x \in S^n : \varphi(x) = x \}$$
$$= \{ (x^1, x^2, x^3, 0, \dots, 0) : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \}$$
$$= S^2 \text{ is a totally geodesic submanifold.}$$

Hence $K_{S^n}(\Pi) = K_{S^2}(\Pi) = +1$,

$\forall$ 2-plane $\Pi \subset T_x S^2 \subset T_x S^n$ (where $x = (x^1, x^2, x^3, 0, \dots, 0)$)

Repeat the argument for any 3 indices $i, j, k \in \{1, \dots, n+1\}$ and using the fact that S^n is invariant under rotation, we have proved that $K_{\mathbb{S}^n} \equiv +1$. #

<u>Lemma 6</u> $\left(\mathbb{B}^n, \frac{4}{(1-|x|^2)^2} \sum_{i=1}^{n} dx^i \otimes dx^i\right)$, where $|x|^2 = \sum_{i=1}^{n} (x^i)^2$,

is a <u>complete</u> Riemannian metric with constant sectional curvature -1.

<u>Pf</u>: (1) <u>Completeness</u>

<u>Pf</u>: First note that $\forall A \in O(n)$

$$A\big|_{\mathbb{B}^n} : \mathbb{B}^n \to \mathbb{B}^n \text{ is an isometry of}$$

the hyperbolic geometry.

(since A preserves $|x|$ & $\sum dx^i \otimes dx^i$)

Now consider the curve

$$\begin{aligned} \zeta(s) : (-\infty, \infty) &\longrightarrow \mathbb{B}^n \\ s &\longmapsto \left(\frac{e^s-1}{e^s+1}, 0, \cdots, 0\right) \end{aligned}$$

Then $\zeta'(s) = \left(\frac{2e^s}{(e^s+1)^2}, 0, \cdots, 0\right)$

$$\Rightarrow \quad |\zeta'(s)|^2_{hyp} = \frac{4}{(1-|\zeta(s)|^2)^2} |\zeta'(s)|^2_{Eucl} \overset{(Ex)}{=\!=} 1.$$

Let $A \in O(n)$ be given by

$$A(x^1, x^2, \cdots, x^n) = (x^1, -x^2, \cdots, -x^n)$$

Then $\zeta((-\infty,\infty)) = \{x \in \mathbb{B}^n : Ax = x\}$

$$= \{(x^1, 0, \cdots, 0) : -1 < x^1 < 1\}$$

Lemma 4 $\Rightarrow$ ζ is a normalized geodesic defined on the whole $(-\infty,\infty)$ with $\zeta'(0)$ in the e_1-direction ($\{e_i\}$ = standard basis of $\mathbb{R}^n$)

Applying other $A \in O(n)$, we have geodesic with

$(A\zeta)'(0)$ = any given direction, and

$A\zeta(s)$ defined on the whole $(-\infty,\infty)$.

Therefore $\exp_0$ is defined on the whole $T_0\mathbb{B}^n$.

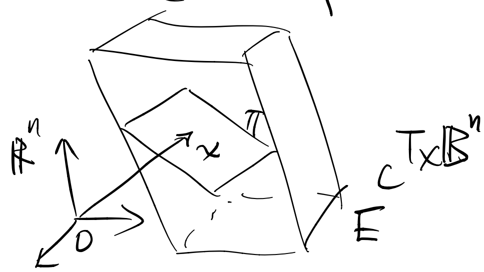
Hence Hopf-Rinow $\Rightarrow$ $\mathbb{B}^n$ is complete.

(2) Curvature $\equiv -1$.

Pf: Let $x \in \mathbb{B}^n$ and $\Pi \subset T_x\mathbb{B}^n$ be a 2-plane.

Identify $T_x\mathbb{B}^n \cong \mathbb{R}^n$

and x can be considered as an



element in $\mathbb{R}^n$. Assume $n \geq 3$.

Take a 3-dim'l subspace $E \subset \mathbb{R}^n$ s.t.

$$\mathrm{span}\{x, \Pi\} \subset E$$

(If $x \neq 0$ & $x \notin \Pi$, then E is unique, otherwise not)

Then $\mathbb{R}^n = E \oplus E^\perp$ orthogonal (in Euclidean)

and one can defines a map

$$\phi : (e, e') \mapsto (e, -e'), \quad e \in E, \ e' \in E^\perp$$

Then $\phi|_{\mathbb{B}^n}$ is an isometry of $\mathbb{B}^n$ with fixed point set

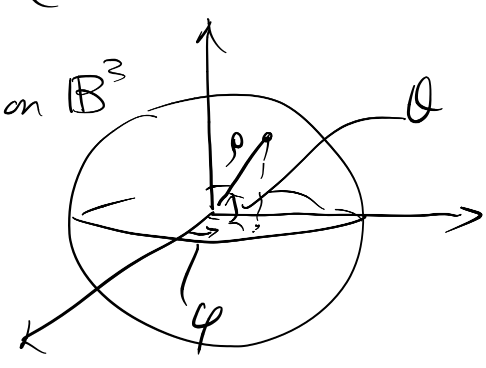
$E \cap \mathbb{B}^n$.

$\Rightarrow$ $\mathbb{B}^3 = E \cap \mathbb{B}^n$ is a totally geodesic submanifold of $\mathbb{B}^n$

$\Rightarrow$ $K_{\mathbb{B}^n}(\Pi) = K_{\mathbb{B}^3}(\Pi)$.

So we only need to show the case that $n = 3$.

Let $\{\rho, \varphi, \theta\}$ = polar coordinates on $\mathbb{B}^3$



$\Rightarrow$ on $\mathbb{B}^3 \setminus \{0\}$, the metric

$\frac{4}{(1-|x|^2)^2} \sum dx^i \otimes dx^i$ can be written as

$$\frac{4}{(1-\rho^2)^2}\left(d\rho^2+\rho^2 d\theta^2+\rho^2\cos^2\theta\, d\varphi^2\right).$$

(where $d\rho^2 = d\rho\otimes d\rho$ ...)

Let
$$\begin{cases} e_1 = \dfrac{1-\rho^2}{2}\dfrac{\partial}{\partial\rho} \\[2ex] e_2 = \dfrac{1-\rho^2}{2\rho}\dfrac{\partial}{\partial\theta} \\[2ex] e_3 = \dfrac{1-\rho^2}{2\rho\cos\theta}\dfrac{\partial}{\partial\varphi} \end{cases}$$

Then $\langle e_i, e_j\rangle = \delta_{ij}$ (Ex.)

$$\Rightarrow \langle D_{e_i}e_j, e_k\rangle = \frac{1}{2}\Big\{\cancel{e_i\langle e_j,e_k\rangle} + \cancel{e_j\langle e_k,e_i\rangle} - \cancel{e_k\langle e_i,e_j\rangle} + \langle e_k,[e_i,e_j]\rangle + \langle e_j,[e_k,e_i]\rangle - \langle e_i,[e_j,e_k]\rangle\Big\}$$

$$= \frac{1}{2}\Big\{\langle e_k,[e_i,e_j]\rangle + \langle e_j,[e_k,e_i]\rangle - \langle e_i,[e_j,e_k]\rangle\Big\}$$

Now $[e_1,e_2] = \dfrac{1-\rho^2}{2}\dfrac{\partial}{\partial\rho}\left(\dfrac{1-\rho^2}{2\rho}\dfrac{\partial}{\partial\theta}\right) - \dfrac{1-\rho^2}{2\rho}\dfrac{\partial}{\partial\theta}\left(\dfrac{1-\rho^2}{2}\dfrac{\partial}{\partial\rho}\right)$

$$= \frac{1-\rho^2}{2}\left(\frac{1-\rho^2}{2\rho}\right)' \frac{\partial}{\partial\theta} = -\frac{1+\rho^2}{2\rho} e_2 \quad (\text{Ex.})$$

Similarly

$$\left.\begin{aligned} [e_2, e_3] &= \frac{1-\rho^2}{2\rho}\tan\theta \; e_3 \\ [e_1, e_3] &= -\frac{1+\rho^2}{2\rho} e_3 \end{aligned}\right\} (\text{Ex.})$$

Then straight forward calculation (Ex.) $\Rightarrow$

$$\begin{cases} D_{e_1}e_1 = 0, \quad D_{e_2}e_1 = \frac{1+\rho^2}{2\rho} e_2, \quad D_{e_3}e_1 = \frac{1+\rho^2}{2\rho} e_3 \\ D_{e_1}e_2 = 0, \quad D_{e_2}e_2 = -\frac{1+\rho^2}{2\rho} e_1, \quad D_{e_3}e_2 = -\frac{1-\rho^2}{2\rho}\tan\theta \; e_3 \\ D_{e_1}e_3 = 0, \quad D_{e_2}e_3 = 0, \quad D_{e_3}e_3 = -\frac{1+\rho^2}{2\rho} e_1 + \frac{1-\rho^2}{2\rho}\tan\theta \; e_2 \end{cases}$$

Hence

$$\begin{aligned} R(e_1,e_2,e_1,e_2) &= \langle R_{e_1e_2}e_1, e_2\rangle \\ &= \langle D_{[e_1,e_2]}e_1 - [D_{e_1}, D_{e_2}]e_1, e_2\rangle \end{aligned}$$

$$= -\frac{1+\rho^2}{2\rho}\langle D_{e_2}e_1, e_2\rangle - \langle D_{e_1}(D_{e_2}e_1) - D_{e_2}(\cancel{D_{e_1}}e_1), e_2\rangle$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2\langle e_2, e_2\rangle - \langle D_{e_1}\left(\frac{1+\rho^2}{2\rho} e_2\right), e_2\rangle$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 - e_1\left(\frac{1+\rho^2}{2\rho}\right)$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 - \frac{1-\rho^2}{2}\left(\frac{1+\rho^2}{2\rho}\right)'$$

$$= -1 \quad (\text{Ex.})$$

Similarly $R(e_1,e_3,e_1,e_3) = R(e_2,e_3,e_2,e_3) = -1$ (Ex!)

To complete the proof, we need to show that all other (Ex.)

$$R(e_i,e_j,e_k,e_l) = 0$$

Since $n=3$, the indices have to be repeated.

It is clear that if $i=j=k=l$ or 3 of the indices are equal, then $R(e_i,e_j,e_k,e_l)=0$

Therefore, we only need to consider

$$R(e_i,e_j,e_i,e_k) \quad \text{with } j<k \quad (i,j,k \text{ distinct})$$

Other cases are clear zero or can be reduced to this case.
(If $j=k$, this is the previous situation)

For $i=3$

$$R(e_3,e_1,e_3,e_2) = \langle R_{e_3 e_1} e_3, e_2 \rangle$$

$$= \langle D_{[e_3,e_1]} e_3, e_2 \rangle - \langle D_{e_3} D_{\cancel{e_1}} e_3, e_2 \rangle + \langle D_{e_1} D_{e_3} e_3, e_2 \rangle$$

$$= \frac{1+\rho^2}{2\rho} \langle D_{e_3} e_3, e_2 \rangle + \langle D_{e_1}(D_{e_3} e_3), e_2 \rangle$$

$$= \frac{1+\rho^2}{2\rho} \cdot \frac{1-\rho^2}{2\rho} \tan\theta + \left\langle D_{e_1}\left(-\frac{1+\rho^2}{2\rho} e_1 + \frac{1-\rho^2}{2\rho}\tan\theta\, e_2\right), e_2 \right\rangle$$

$$= \frac{1-\rho^4}{4\rho^2}\tan\theta + e_1\left(\frac{1-\rho^2}{2\rho}\tan\theta\right)$$

$$= \frac{1-\rho^4}{4\rho^2}\tan\theta + \frac{1-\rho^2}{2}\left(\frac{1-\rho^2}{2\rho}\right)'\tan\theta$$

$$= 0 \quad (\text{Ex.})$$

Similarly $R(e_1, e_2, e_1, e_3) = R(e_2, e_1, e_2, e_3) = 0$. (Ex).

Hence $\mathbb{B}^3$ has sectional curvature $\equiv -1$. ※

Existence of Thm 1: By Lemma 5 and Lemma 6, we have complete simply-connected Riemannian manifolds of any dimension ≥ 2 with constant sectional curvature $= \pm 1$.

By scaling, we have $K_{\frac{1}{c}g} = cK_g$ ($\forall$ metric g (Ex))

$$= \pm c$$

Together with $\mathbb{R}^n$, we've proved the existence part of Thm 1. ※