Pf of 2nd Bianchi Identity:

It is sufficient to prove the identity for vector fields satisfying $[X, Y] = \cdots = 0$.

For these vector fields $\begin{cases} D_X Y = D_Y X \\ R_{XY} = -D_X D_Y + D_Y D_X \end{cases}$

By definition

$$(D_X R)_{YZ} W = D_X(R_{YZ} W) - R_{(D_X Y) Z} W - R_{Y (D_X Z)} W - R_{YZ}(D_X W)$$

$$(D_Y R)_{ZX} W = D_Y(R_{ZX} W) - R_{(D_Y Z) X} W - R_{Z (D_Y X)} W - R_{ZX}(D_Y W)$$

$$(D_Z R)_{XY} W = D_Z(R_{XY} W) - R_{(D_Z X) Y} W - R_{X (D_Z Y)} W - R_{XY}(D_Z W)$$

$$\Rightarrow (D_X R)_{YZ} W + (D_Y R)_{ZX} W + (D_Z R)_{XY} W$$

$$= D_X(-\overset{1}{\cancel{D_Y}} D_Z W + \overset{2}{\cancel{D_Z}} D_Y W) + D_Y(-\overset{3}{\cancel{D_Z}} D_X W + \overset{4}{\cancel{D_X}} D_Z W)$$
$$+ D_Z(-\overset{5}{\cancel{D_X}} D_Y W - \overset{6}{\cancel{D_Y}} D_X W) - (\overset{6}{\cancel{D_Z}} D_Y - \overset{3}{\cancel{D_Y}} D_Z)(D_X W)$$

$$-(\overset{2}{\cancel{D_X D_Z}} - \overset{5}{\cancel{D_Z D_X}})(D_Y W) - (\overset{4}{\cancel{D_Y D_X}} - \overset{1}{\cancel{D_X D_Y}})(D_Z W)$$

$$-\overset{a}{\cancel{R_{(D_X Y) Z}}} W - \overset{b}{\cancel{R_{Y (D_X Z)}}} W - \overset{c}{\cancel{R_{(D_Y Z) X}}} W - \overset{a}{\cancel{R_{Z (D_Y X)}}} W$$

$$-\overset{b}{\cancel{R_{(D_Z X) Y}}} W - \overset{c}{\cancel{R_{X (D_Z Y)}}} W$$

$$= 0$$

<u>Pf of Ricci Identity</u>

$$D^2 T(\cdots, X, Y)$$

$$= D(DT)(\cdots, X, Y)$$

$$= \left(D_Y (DT)\right)(\cdots, X)$$

$$= D_Y\left((DT)(\cdots, X)\right) - \sum (DT)(\cdot\cdot D_Y \cdot \cdot\cdot, X)$$
$$- (DT)(\cdots, D_Y X)$$

$$= D_Y\left((D_X T)(\cdots)\right) - \sum (D_X T)(\cdot\cdot D_Y \cdot \cdot\cdot)$$
$$- \left(D_{(D_Y X)} T\right)(\cdots)$$

$$= \left(D_Y D_X T\right)(\cdots) - \left(D_{D_Y X} T\right)(\cdots)$$

$$= \left(D_Y D_X T - D_{D_Y X} T\right)(\cdots)$$

Similarly for $D^2T(\cdots, Y, X)$

Hence

$$(D^2T)(\cdots, X, Y) - D^2T(\cdots, Y, X)$$

$$= \left(D_Y D_X T - D_{D_Y X} T - D_X D_Y T + D_{D_X Y} T\right)(\cdots)$$

$$= \left[\left(-D_X D_Y + D_Y D_X + D_{[X,Y]}\right)T\right](\cdots)$$

$$= (R_{XY} T)(\cdots) \quad \#$$

<u>Note</u>: So we usual denote Ricci Identity by

$$\boxed{R_{XY} = D^2_{XY} - D^2_{YX}}$$

3.4 <u>Various notions of curvature</u>

<u>Def</u>: The <u>Ricci tensor</u> "Ric" is the $(0,2)$-tensor field defined by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} R(e_i, X, e_i, Y), \quad \forall X, Y \in \Gamma(TM)$$

where $\{e_i\}$ = orthonormal basis of T_xM.

<u>Note</u>: • Ric doesn't depend on the o.n. basis $\{e_i\}$

- Ric is symmetric, i.e. $\mathrm{Ric}(X,Y)=\mathrm{Ric}(Y,X)$.

Def: Let $X\in T_xM$ with $|X|=1$. Then $\mathrm{Ric}(X,X)$ is called the Ricci curvature in the direction of X.

Note: One can choose an o.n. basis $\{e_1,\cdots,e_n\}$ of T_xM such that $e_1=X$. Then by definition of Ric

$$\mathrm{Ric}(X)\overset{\text{def}}{=}\mathrm{Ric}(X,X)=\sum_{i=1}^{n}R(e_i,X,e_i,X)$$

$$=\sum_{i=1}^{n}R(e_i,e_1,e_i,e_1)$$

$$=\sum_{i=2}^{n}R(e_i,e_1,e_i,e_1)$$

$$=\sum_{i=2}^{n}K(\Pi_i)$$

where $\Pi_i=\mathrm{span}\{e_1,e_i\}$.

Def: The scalar curvature $s(x)$ at $x\in M$ is defined by

$$s(x)=\sum_{i,j}R(e_i,e_j,e_i,e_j)$$

where $\{e_1,\cdots,e_n\}$ = o.n. basis of T_xM.

i.e. scalar curvature = sum of all sectional curvatures of planes given by an o.n. basis.

Ch4 Exponent Map, Gauss Lemma, & Completeness

Let
- M = Riemannian manifold with metric
- $g = g_{ij}\, dx^i \otimes dx^j \quad (g = \langle\, ,\, \rangle)$
- D = Levi-Civita connection of g

4.1 Exponent map

Recall: $\gamma : [0, L] \to M$ is a geodesic (wrt D)

$$\Leftrightarrow \quad D_{\gamma'}\gamma' = 0 .$$

Facts:
- If γ is a geodesic, $|\gamma'|$ is a constant.
- If $\gamma : [0, L] \to M$ is a geodesic, then $\forall$ constant $c > 0$,

$$\gamma^c : [0, \tfrac{L}{c}] \to M$$
$$t \mapsto \gamma(ct)$$

is also a geodesic, and

$$|(\gamma^c)'| = c|\gamma'|$$

Therefore, we can normalize our geodesic to have $|\gamma'| = 1$.

Recall: If $\xi : [a, b] \to M$ is a C^∞ curve, then the length of ξ is defined by

$$L(\xi) = \int_a^b |\xi'| dt .$$

If ξ is regular, ie $|\xi'(t)| > 0$, $\forall t \in [a,b]$,

then $$s(t) = \int_a^t |\xi'(\tau)| d\tau = L(\xi|_{[a,t]})$$

defines a C^∞ function $s = [a,b] \to [0, L(\xi)]$

with $\frac{ds}{dt} = |\xi'(\tau)| > 0.$

Hence $u = s^{-1} = [0, L(\xi)] \to [a,b]$ exists & C^∞.

And $\tilde{\xi}(s) \overset{def}{=} \xi(u(s)) : [0, L(\xi)] \to M$ is

a reparametrization of ξ such that

$$\left| \frac{d\tilde{\xi}}{ds} \right| = 1 .$$

Terminology:
- s = arc-length parameter
- $\tilde{\xi}$ is said to be parametrized by arc-length.
- a normalized geodesic is a geodesic parametrized by arc-length

ie. $D_{\gamma'} \gamma' = 0$ and $|\gamma'| = 1$.

Note: All the above can be extended to piecewise C^1 curves.

Recall: $D_{\gamma'}\gamma' = 0$ is a (nonlinear) ODE system and hence we have the following result by "applying" the theory of ODE:

Thm: $\forall\, x \in M$ & $\varepsilon > 0$,

$\exists$ nbd. $\mathcal{U}$ of x, and $\delta > 0$ such that

$\forall\, y \in \mathcal{U}$ and $v \in T_yM$ with $|v| < \delta$, $\exists$ unique geodesic $\gamma_v : I \to M$, defined on an open interval I containing $[-\varepsilon, \varepsilon]$, with initial condition

$$\begin{cases} \gamma_v(0) = y \\ \gamma_v'(0) = v \end{cases}$$

If γ_v is a geodesic by above, then

$$\xi_v(t) \overset{\text{def}}{=\!=} \gamma_v(\varepsilon t) \text{ is a geodesic}$$

defined on an open interval containing $[0, 1]$.

Thm (#) $\forall x \in M$, $\exists$ nbd. $\mathcal{U}$ of x and $\omega > 0$ s.t.

> $\forall y \in \mathcal{U}$ and $v \in T_y M$ with $|v| < \omega$, $\exists$ unique geodesic $\gamma_v : I \to M$ defined on an open interval I containing $[0,1]$ with initial conditions
> $$\gamma_v(0) = y \quad \text{and} \quad \gamma_v'(0) = v.$$

Def: Let $\omega > 0$ be given in Thm(#). The exponential map $\exp_x$ at x, defined on

$$B_x(\omega) = \{ v \in T_x M : |v| < \omega \} \subset T_x M$$

is the map

$$\begin{array}{rcl} \exp_x : B_x(\omega) & \longrightarrow & M \\ \Psi & & \Psi \\ v & \longmapsto & \gamma_v(1) \end{array}$$

where γ_v is given by Thm(#).

That is

$$\boxed{\exp_x(v) = \gamma_v(1)}$$

Fact: Let $\mathcal{V} = \{ (y,v) \in TM : y \in \mathcal{U}, |v| < \omega \} \subset TM$ (with $\mathcal{U}$ & ω as in Thm(#)). Then Thm(#)

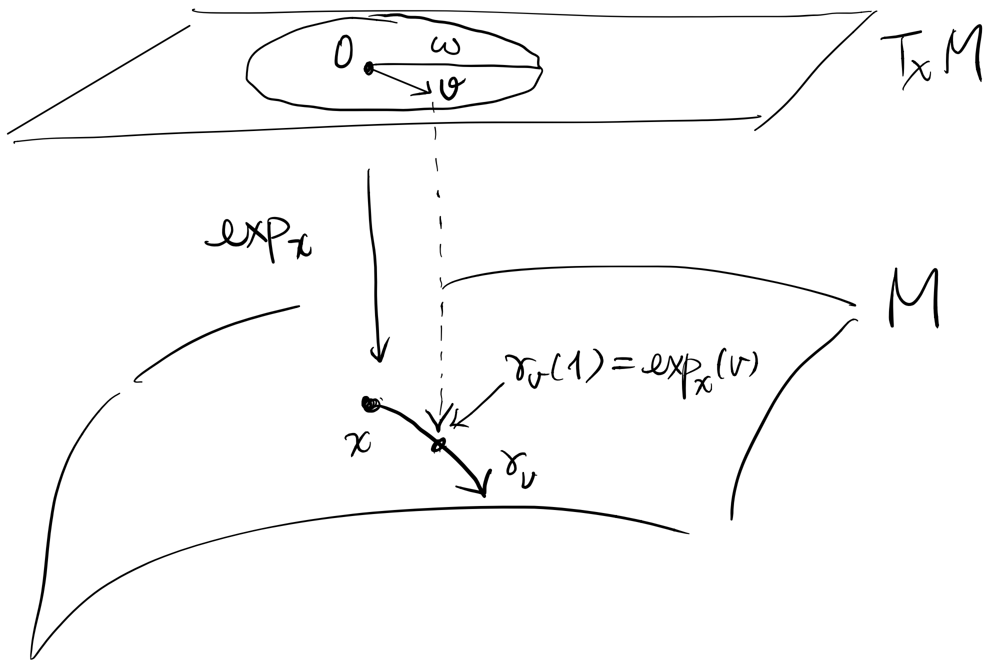
$\Rightarrow \exp(y,v) \stackrel{\text{def}}{=} \exp_y(v)$ defines a map

$\exp : \mathcal{U} \to M$. By ODE theory & Thm (#),

$\exp : \mathcal{U} \to M$ is C^∞ & in particular

$\exp_x : B_x(\omega) \to M$ is C^∞

(Pf : See Gallot, Hulin, & Lafontaine)



Note : In fact, we can show that

$$\exp_x : \mathcal{B} \to M \in C^\infty$$

on the maximal domain of the definition of $\exp_x$.

Note : In the case of

$$M = SO(n, \mathbb{R}) = \{ A \in n\times n\text{-matrix} : A^T A = I,\ \det A = 1 \}$$

with metric defined by $(n-2)\,tr(XY)$ for

$X, Y \in so(n, \mathbb{R}) = T_{Id}M = \{ B \in n\times n\text{-matrix} : B^T + B = 0 \}$

Then $\exp_{Id} : T_{Id}M \to M$ is given by

$$\exp_{Id} B = e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}$$

$\forall\, B \in T_{Id}M = \{ B^T + B = 0 \} = so(n, \mathbb{R})$.

This is the reason for the terminology.

<u>Thm</u>: $\exp_x$ is a diffeomorphism in a nbd of $0 \in T_xM$.

This Thm follows immediately from

<u>Lemma</u>: $(d\exp_x)_0 =$ "identity of T_xM".

<u>Note</u>: $\exp_x : B(\omega) \subset T_xM \longrightarrow M$ with $\exp_x(0) = x$.

Therefore $(d\exp_x)_0 : T_0(T_xM) \to T_xM$

Since T_xM is linear, $T_0(T_xM) \cong T_xM$.

In fact, $\forall\, v \in T_xM$, we define

$\xi_v : t \mapsto tv$ a curve in T_xM

with $\xi_v(0) = 0$, "$\xi_v'(0) = v$"

Hence $(d\exp_x)_0$ can be regarded as a map from T_xM to itself.

<u>Pf of Lemma</u> : $\forall\, v \in T_xM \cong T_0(T_xM)$

$$(d\exp_x)_0(v) = \left.\frac{d}{dt}\right|_{t=0} \exp_x(tv) \qquad \left(\begin{array}{c} v \leftrightarrow [tv] \\ \rotatebox{90}{$\in$} \\ T_0(T_xM) \end{array} \right)$$

$$= \left.\frac{d}{dt}\right|_{t=0} \gamma_{tv}(1)$$

$$= \left.\frac{d}{dt}\right|_{t=0} \gamma_v(t)$$

$$= \gamma_v'(0) = v. \qquad \#$$

We can even prove a stronger result:

<u>Thm</u> : $\forall$ compact $K \subset M$, $\exists\, \varepsilon > 0$ s.t.

$$\forall x \in K,\ \exp_x \text{ is diffeo. on } B_x(\varepsilon) = \{v \in T_xM : |v| < \varepsilon\}.$$

(This shows that for any fixed cpt K, we can find a uniform $\varepsilon > 0$)

<u>Pf</u> : It is sufficient to show that

$$\boxed{\begin{array}{l} \forall x \in M,\ \exists\, \varepsilon > 0,\ \&\text{ open nbd } \Omega \text{ of } x \text{ s.t.} \\ \forall y \in \Omega,\ \exp_y \text{ is a diffeo. on } B_y(\varepsilon) \subset T_yM. \end{array}}$$

By Thm(#), $\exists$ nbd $\mathcal{U}$ of x s.t. $\exp_y$ is defined on some ball $B_y(\varepsilon(y))$, $\varepsilon(y) > 0$.

Let $N = \{(y,v) : y \in \mathcal{U},\ v \in B_y(\varepsilon(y))\} \subset TM$

and define

$$E : \begin{array}{ccc} N & \longrightarrow & M \times M \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ (y,v) & \longmapsto & (y, \exp_y v) \end{array}$$

By the theory of ODE, E is C^∞.

Choose a coordinate system $\{x^1, \cdots, x^n\}$ centered at x (i.e. $x^i(x) = 0$, $\forall i = 1, \cdots, n$). Then for any (y,v), we can represent it by coordinates

$$(x^1, \cdots, x^n, u^1, \cdots, u^n)$$

where $\{u^i\}$ are given by $v = \sum u^i \frac{\partial}{\partial x^i}$.

(i.e. $u^i = dx^i(v)$, $\forall i = 1, \cdots, n$)

$\Rightarrow$ $\left\{ \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial u^1}, \cdots, \frac{\partial}{\partial u^n} \right\}$ is a basis of the tangent space $T_{(y,v)}(TM)$ of TM.

Now

$$dE_{(x,0)}\left(\frac{\partial}{\partial x^i}\Big|_{(x,0)}\right) = \frac{d}{dt}\Big|_{t=0} E(\xi_i(t), 0)$$

where $\xi_i(t)$ is a curve in M s.t.

$$\xi_i(0) = x \quad \& \quad \xi_i'(0) = \frac{\partial}{\partial x^i}\Big|_x .$$

$$\Rightarrow dE_{(x,0)}\left(\frac{\partial}{\partial x^i}\Big|_{(x,0)}\right) = \frac{d}{dt}\Big|_{t=0} (\xi_i(t), \exp_{\xi_i(t)} 0)$$

$$= \frac{d}{dt}\Big|_{t=0} (\xi_i(t), \xi_i(t))$$

$$= \left(\frac{\partial}{\partial x^i}\Big|_x , \frac{\partial}{\partial x^i}\Big|_x\right)$$

Also $dE_{(x,0)}\left(\frac{\partial}{\partial u^i}\Big|_{(x,0)}\right) = \frac{d}{dt}\Big|_{t=0} E\left(x, t\frac{\partial}{\partial x^i}\Big|_x\right)$

$$= \frac{d}{dt}\Big|_{t=0}\left(x, \exp_x\left(t\frac{\partial}{\partial x^i}\Big|_x\right)\right)$$

$$= \left(0, (d\exp_x)_0\left(\frac{\partial}{\partial x^i}\Big|_x\right)\right)$$

$$= \left(0, \frac{\partial}{\partial x^i}\Big|_x\right) \quad \text{by previous lemma.}$$

$\Rightarrow dE_{(x,0)} : T_{(x,0)}N \longrightarrow T_xM \times T_xM$

is nonsingular. $\therefore$ IFT $\Rightarrow E$ is a local diffeo that maps a nbd $\mathcal{W}$ of $(x,0)$ in TM to a nbd of $(x, \exp_x(0)) = (x,x)$ in $M\times M$.

Therefore, $\exists\, c>0$, $\varepsilon'>0$ such that

$$\{(y,v)\in TM : |x^i(y)|\le c,\ |u^i(v)|\le \varepsilon'\}$$

is a cpt. subset of $\mathcal{W}$.

norm wrt metric g

$\Rightarrow \exists\, \varepsilon>0$ s.t.

$$\{(y,v)\in TM : |x^i(y)|\le c,\ |v|\le \varepsilon\} \subset \mathcal{W}.$$

Then this $\varepsilon>0$ & $\Omega = \{y\in \mathcal{U} : |x^i(y)|\le c\}$ satisfy the requirement. $\#$