Pf of (3) We do only the special case that

$$K = \mathbf{X} \otimes \rho \quad \in TM \otimes T^*M \quad \text{and}$$

$$\begin{array}{cccc} \mathcal{C} : & TM \otimes T^*M & \longrightarrow & \mathbb{R} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \mathbf{X} \otimes \rho & \longmapsto & \rho(\mathbf{X}) \end{array}$$

In this case
$$\mathcal{C}K = \mathcal{C}(\mathbf{X} \otimes \rho) = \rho(\mathbf{X})$$

$$D_v(\mathcal{C}K) = v(\rho(\mathbf{X}))$$

$$\begin{aligned} \mathcal{C}(D_v K) &= \mathcal{C}(D_v(\mathbf{X} \otimes \rho)) \\ &= \mathcal{C}(D_v \mathbf{X} \otimes \rho + \mathbf{X} \otimes D_v \rho) \\ &= \rho(D_v \mathbf{X}) + (D_v \rho)(\mathbf{X}) \end{aligned}$$

Note that
$$\left\{ \begin{aligned} \rho(\mathbf{X}) &= \Big(\sum_l \rho_l \alpha^l(t)\Big)\Big(\sum_i \mathbf{X}^i e_i(t)\Big) \\ &= \sum_i \rho_i \mathbf{X}^i \\ \rho(D_v \mathbf{X}) &= \sum_i \rho_i \frac{d\mathbf{X}^i}{dt} \\ (D_v \rho)(\mathbf{X}) &= \sum_i \frac{d\rho_i}{dt} \mathbf{X}^i \end{aligned} \right.$$

$$\therefore \quad v(\rho(\mathbf{X})) = v\Big(\sum_i \rho_i \mathbf{X}^i\Big) = \sum_i \Big(\rho_i \frac{d\mathbf{X}^i}{dt} + \frac{d\rho_i}{dt} \mathbf{X}^i\Big)$$

$$= \rho(D_v X) + (D_v \rho)(X) \quad . \#$$

<u>Note that</u> one can define $D_v \rho$ by this special case:

$$\boxed{(D_v \rho)(X) = v(\rho(X)) - \rho(D_v X), \quad \forall X \in \Gamma(M)}$$

- This also show that $D_v K$ does not depend on curve γ in the definition.

<u>Def</u>: Let K = tensor field on M

X = vector field on M.

Then we define $(D_X K)(x) \overset{\text{def}}{=\!=} D_{X(x)} K$, $\forall x \in M$.

<u>Note</u>: By linearity of $D_X K$ in X, one can define

$$DK \in \left(\otimes^r TM\right) \otimes \left(\otimes^{s+1} T^*M\right)$$

$$\left(\text{for } K \in \left(\otimes^r TM\right) \otimes \left(\otimes^s T^*M\right)\right)$$

by requiring

$$(DK)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s \otimes X)$$

$$\stackrel{\text{def}}{=} (D_X K)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s)$$

(Caution: Some authors put

$$(DK)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X \otimes X_1 \otimes \cdots \otimes X_s)$$
$$= (D_X K)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s)$$
)

Note: If $K = f \in T^{(0,0)}M \cong C^\infty(M)$.
Then $Df = df$ the usual differential of f.
(Check!)

Def: For $n \geq 0$, we define

$$D^{n+1}K = D(D^n K)$$

Note: $(D^2 K)(\cdots, X, Y) \neq (D_Y(D_X K))(\cdots)$
in general.

eg: Let $K = f \in C^\infty(M)$.
Then
$$\begin{aligned}(D^2 f)(X, Y) &= (D\, df)(X, Y) \\ &= (D_Y\, df)(X) \\ &= Y(df(X)) - df(D_Y X)\end{aligned}$$

$$= Y(X(f)) - (D_Y X)(f)$$

$$\neq D_Y(D_X f)$$

(by definition $D_Y(D_X f) = D_Y(X(f)) = Y(X(f))$)

<u>Note</u>: $\begin{cases} (D^2 f)(X,Y) = YXf - (D_Y X)f \\ (D^2 f)(Y,X) = XYf - (D_X Y)f \end{cases}$

$$\Rightarrow \quad (D^2 f)(X,Y) - D^2 f(Y,X)$$

$$= -[X,Y]f + (D_X Y - D_Y X)f$$

$$= T(X,Y)f$$

$\uparrow$ torsion tensor.

$\therefore$ D symmetric (torsion free) $\Leftrightarrow$ $D^2 f$ is symmetric

In this case, $D^2 f$ is called the <u>Hessian</u> of f.

<u>From now on</u>, we assume M has a Riemannian metric g and D = Levi-Civita connection of g.

Therefore $D^2 f$ is always symmetric for $f \in C^\infty(M)$.

Def: $\forall\, S \in \otimes^2 T^*M$, we define $\operatorname{tr} S \in C^\infty(M)$ the trace of S, by

$$\operatorname{tr} S(x) = \sum_i S(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal basis of T_xM.

(Check: (i) $\operatorname{tr} S$ is well-defined, i.e. independent of the choice of the o.n. basis $\{e_i\}$.

(ii) $\operatorname{tr} S(x)$ is smooth in x.)

Def: Let (M, g) = Riemannian manifold

D = Levi-Civita connection of g

Then the Laplace operator, Laplacian or Laplace-Beltrami operator

$$\Delta : C^\infty(M) \longrightarrow C^\infty(M)$$

is defined by

$$\Delta f = \operatorname{tr} D^2 f .$$

Ex: Prove that in local coordinates $(x^1, \cdots, x^n)$

$$\Delta f = \frac{1}{\sqrt{G}} \sum_j \frac{\partial}{\partial x^j}\left(\sum_i g^{ij} \sqrt{G} \frac{\partial f}{\partial x^i} \right)$$

where $G = \det(g_{ij})$, $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ and

$$(g^{ij}) = (g_{ij})^{-1}.$$

3.2 Curvature Tensor

Let $\mathcal{T}^* =$ Algebra of tensor fields on M / $C^\infty(M)$.

Then $\forall$ vector field $X \in \Gamma(M)$,

$$D_X : \mathcal{T}^* \longrightarrow \mathcal{T}^* \quad \text{is a } \underline{\text{derivation}}.$$

Therefore, if we have D_X & D_Y, the Lie bracket

$$[D_X, D_Y] = D_X D_Y - D_Y D_X$$

is also a derivation (Ex.)

Hence we can make the following definition

$$\boxed{R_{XY} = D_{[X,Y]} - [D_X, D_Y]}$$

$$\boxed{= -D_X D_Y + D_Y D_X + D_{[X,Y]}}$$

<u>Prop</u>:

(1) $R_{XY} : \mathcal{T}^* \to \mathcal{T}^*$ is a derivation.

(2) R_{XY} preserves the type of a tensor field

i.e. K is (r,s)-type $\Rightarrow R_{XY} K$ is also (r,s)-type.

(3) $\forall f \in C^\infty(M)$

$$R_{(fX)Y}K = R_{X(fY)}K = R_{XY}(fK) = fR_{XY}K.$$

(4) $\forall f \in C^\infty(M)$, $R_{XY}f = 0$.

Pf: We check only $R_{(fX)Y}K = fR_{XY}K$.

(the others are easy ex!)

$$\begin{aligned}
R_{(fX)Y}K &= -D_{fX}D_Y K + D_Y D_{fX}K + D_{[fX,Y]}K \\
&= -fD_X D_Y K + D_Y(fD_X K) + D_{[fX,Y]}K \\
&= -f D_X D_Y K + f D_Y D_X K + (Yf) D_X K + D_{[fX,Y]}K \\
&= fR_{XY}K - fD_{[X,Y]}K + (Yf)D_X K + D_{[fX,Y]}K.
\end{aligned}$$

Note that
$$\begin{aligned}
[fX, Y] &= (fX)Y - Y(fX) \\
&= f(XY - YX) - (Yf)X
\end{aligned}$$

$$\therefore \quad R_{(fX)Y}K = fR_{XY}K. \quad \#$$

($\therefore$ $D_{[X,Y]}$ is needed in the definition in order to have property (3).)

Note: By property (3), if $K = Z$ is also a vector field, then one can use $R_{XY}Z$ to define a (1,3)-tensor

$$(\omega, X, Y, Z) \xmapsto{R} \omega(R_{XY}Z) \qquad \forall \text{ 1-form } \omega \ \& \ X, Y, Z \in \Gamma(M)$$

It also defines a (0,4)-tensor R (using metric g)

$$R(X,Y,Z,W) = g(R_{XY}Z, W) \quad , \forall X,Y,Z,W \in \Gamma(M)$$

Def: $R_{XY}Z$ or $R(X,Y,Z,W)$ are called the (Riemannian) curvature tensor of g. (More precisely, R is the curvature tensor of g.)

Local formula: In a coordinate system $(x^1, \cdots, x^n)$

if $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \quad \text{(Christoffel symbol)}$$

then $R_{ijkl} \overset{\text{def}}{=\!=} R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})$

is given by

$$\boxed{R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) + \left(g_{rs}\Gamma^r_{jk}\Gamma^s_{il} + g_{rs}\Gamma^r_{jl}\Gamma^s_{ik}\right)}$$

(Pf: Ex!)

Note: (i) $R = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$

(ii) R is a 2nd order non-linear function of g.

Def: Let (M, g) & (N, h) be 2 Riemannian manifolds.

A C^∞ map $\varphi : M \to N$ is called a local isometry

$\Leftrightarrow$ $\forall\, x \in M$,

$$d\varphi : (T_x M, g_x) \to (T_{\varphi(x)} N, h_{\varphi(x)})$$

is an isometry of the inner product spaces.

i.e. $\forall\, v_1, v_2 \in T_x M$,

$$\boxed{h_{\varphi(x)}(d\varphi(v_1), d\varphi(v_2)) = g_x(v_1, v_2)}$$

Note: If φ = local isom. then $\dim M = \dim N$ and φ is an immersion.

Def: $\varphi : (M, g) \to (N, h)$ is called a global isometry, or simply an isometry,

$\iff$ φ is a local isometry and a diffeomorphism.

Fact : Let • $\varphi : (M,g) \to (M',g')$ is an isometry

• D = Levi-Civita connection of g

• D' = " " " " g'

• $X, Y \in \Gamma(M)$ & $X', Y' \in \Gamma(M')$ s.t.

$d\varphi(X) = X'$, $d\varphi(Y) = Y'$.

Then $$d\varphi(D_X Y) = D'_{X'} Y'.$$

$\therefore$ Levi-Civita connection is a metric invariant.

(Pf: Ex!)

Thm (Metric invariance of curvature tensor)

Let • $\varphi : (M,g) \to (M',g')$ is an isometry

• R, R' = curvature tensors of g & g' respectively

• $X, Y, Z, W \in \Gamma(M)$, $X', Y', Z', W' \in \Gamma(M')$

s.t. $d\varphi(X) = X'$, $d\varphi(Y) = Y'$, $d\varphi(Z) = Z'$, $d\varphi(W) = W'$.

Then

- $$d\varphi(R_{XY}Z) = R'_{X'Y'}Z'$$
- $$R(X,Y,Z,W) = R'(X',Y',Z',W')\circ\varphi.$$

(Pf: Ex!)

__Note__: If $\dim M = 2$, then one can define the __Gaussian curvature__ $K : M \to \mathbb{R}$ by

$$K(x) = R(e_1, e_2, e_1, e_2)(x), \quad \forall x \in M$$

for any __orthonormal basis__ $\{e_1, e_2\}$ of T_xM.

And this K __coincides__ with the original definition for $M^2 \subset \mathbb{R}^3$.

__Def__: A Riemannian manifold (M, g) is called __flat__ if its curvature tensor $R \equiv 0$.

__eg__ $(\mathbb{R}^n, \text{standard metric}) = (\mathbb{R}^n, dx^1\otimes dx^1 + \dots + dx^n \otimes dx^n)$ is flat. (Reason: $g_{ij} = \text{const.} \Rightarrow \Gamma_{ij}^k = 0 \Rightarrow R = 0$)

3.3 Basic properties of curvature tensor

Lemma 1 $\forall$ vector fields X, Y, Z, W

(1) $R_{XY} = - R_{YX}$

(2) (1st Bianchi identity)

$$R_{XY}Z + R_{YZ}X + R_{ZX}Y = 0$$

(3) $R(X,Y,Z,W) = - R(X,Y,W,Z)$

(4) $R(X,Y,Z,W) = R(Z,W,X,Y)$

Pf: (1) is clear.

For (2) and (3), we only need to check the case that $\{X,Y,Z,W\} = \{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\}$.

(Since R is a tensor.)

In this case, $0 = [X,Y] = \cdots = [Z,W]$

Hence $\begin{cases} D_X Y = D_Y X \\ R_{XY} = - D_X D_Y + D_Y D_X \end{cases}$

$\Rightarrow R_{XY}Z + R_{YZ}X + R_{ZX}Y$

$$= (-D_X D_Y Z + D_Y D_X Z) + (-D_Y D_Z X + D_Z D_Y X) + (-D_Z D_X Y + D_X D_Z Y)$$

$$= D_X(\cancelto{0}{-D_Y Z + D_Z Y}) + D_Y(\cancelto{0}{D_X Z - D_Z X}) + D_Z(\cancelto{0}{D_Y X - D_X Y}).$$

$$= 0$$

This proves (2).

For (3), we first note that

$$\begin{aligned}
R(X,Y,Z,Z) &= \langle R_{XY} Z, Z\rangle \\
&= \langle -D_X D_Y Z + D_Y D_X Z, Z\rangle \\
&= -X\langle D_Y Z, Z\rangle + \langle D_Y Z, D_X Z\rangle \\
&\qquad + Y\langle D_X Z, Z\rangle - \langle D_X Z, D_Y Z\rangle \\
&= -\frac{1}{2} X\left(Y\langle Z,Z\rangle\right) + \frac{1}{2} Y\left(X\langle Z,Z\rangle\right) \\
&= -\frac{1}{2}[X,Y]\langle Z,Z\rangle = 0.
\end{aligned}$$

Hence for any $\{X,Y,Z,W\}$ with $[X,Y]=0$,

we have
$$\begin{aligned}
0 &= R(X,Y,Z+W,Z+W) \\
&= \cancelto{0}{R(X,Y,Z,Z)} + R(X,Y,Z,W) + R(X,Y,W,Z) \\
&\qquad + \cancelto{0}{R(X,Y,W,W)}.
\end{aligned}$$

This proves (3).

Proof of (4) (Jost)

$$R(X,Y,Z,W) = -R(Y,X,Z,W) \quad \text{by (1)}$$

$$= R(Z,Y,X,W) + R(X,Z,Y,W)$$

(1st Bianchi)

Similarly

$$R(X,Y,Z,W) = -R(X,Y,W,Z)$$

$$= R(Y,W,X,Z) + R(W,X,Y,Z)$$

$\Rightarrow$

$$2R(X,Y,Z,W) = R(Z,Y,X,W) + R(X,Z,Y,W) + R(Y,W,X,Z) + R(W,X,Y,Z) \quad \text{—} (*)$$

Similarly

$$2R(Z,W,X,Y) = R(X,W,Z,Y) + R(Z,X,W,Y) + R(W,Y,Z,X) + R(Y,Z,W,X)$$

(by (1) & (3)) $= R(W,X,Y,Z) + R(X,Z,Y,W) + R(Y,W,X,Z) + R(Z,Y,X,W)$

(by $(*)$) $= 2R(X,Y,Z,W)$ ※

<u>Lemma 2</u> Let $Q(X,Y) \stackrel{\text{def}}{=} R(X,Y,X,Y)$

Then Q determines R.

i.e. if R, R' are tensor fields satisfying (1)–(4) in

Lemma 1, then $Q = Q' \Rightarrow R = R'$.

(Pf : Omitted)

<u>Def</u> : Let
- π be a 2-dimensional subspace in T_xM
- $\{v_1, v_2\}$ = basis of π

Then
$$K(\pi) = \frac{R(v_1, v_2, v_1, v_2)}{|v_1 \wedge v_2|^2}$$

where
$$|v_1 \wedge v_2|^2 = \det(\langle v_i, v_j \rangle) = |v_1|^2 |v_2|^2 - \langle v_1, v_2 \rangle^2 .$$

is called the <u>sectional curvature of π</u>.

<u>Note</u> :
- $K(\pi)$ doesn't depend on the basis $\{v_1, v_2\}$
- If $\{e_1, e_2\}$ = orthonormal basis of π, then
$$K(\pi) = R(e_1, e_2, e_1, e_2)$$
- Lemma 2 $\Rightarrow$ K determines R.
- Sectional curvature K is a metric invariant.

  i.e. If $\varphi : M \to M'$ isometry

  $\pi \in T_xM$, $\pi' \subset T_{\varphi(x)}M'$ are 2-dim'l subspaces with $d\varphi(\Pi) = \Pi'$.

Then $K(\Pi) = K'(\Pi')$.

eg: If $K(\Pi) = 0$, $\forall x \;\&\; \Pi^2 \subset T_xM$, then $R \equiv 0$.

Lemma 3 (The 2^{nd} Bianchi Identity)

$$\boxed{(D_X R)_{YZ} + (D_Y R)_{ZX} + (D_Z R)_{XY} = 0}$$

$\forall X, Y, Z \in \Gamma(TM)$

Lemma 4 (Ricci Identity)

$$\boxed{D^2T(\cdots, X, Y) - D^2T(\cdots, Y, X) = (R_{XY}T)(\cdots)}$$

$\forall$ tensor field T, & $X, Y \in \Gamma(TM)$.