1.7 Partitions of Unity

Recall that all manifolds in this course are supposed to have the property that "partitions of unity" is always possible. That is:

$\forall \{U_i\}_{i\in\Lambda}$ = open cover of M,

$\exists$ locally finite open cover $\{V_k\}_{k\in\Lambda'}$ and a family $\{\varphi_k\}_{k\in\Lambda'}$ of real smooth functions on M such that

- $\{V_k\}_{k\in\Lambda'}$ is subordinate to $\{U_i\}_{i\in\Lambda}$ (i.e. each $V_k \subset U_i$ for some i)
- $\operatorname{supp}\varphi_k \subset V_k$, $\varphi_k \geq 0$, and

$$\sum_{k\in\Lambda'} \varphi_k(x) = 1 \quad \text{for all } x \in M$$

Here $\{V_k\}_{k\in\Lambda'}$ being locally finite means $\forall x \in M$, $\exists$ open nbd W of x such that $W \cap V_k = \emptyset$ except finitely many k's.

Ch 2 Riemannian Metric, Connection and Parallel Transport

§2.1 Riemannian metric & connection

Def: Let M be a C^∞ manifold. A Riemannian metric g on M is given by an inner product g_x on each T_xM which depends smoothly on $x \in M$ in the sense that in any nbd. system U with coordinate functions $x^1, \dots, x^n$,

$$g_{ij}(x) = g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \quad (\forall i, j)$$

is a smooth function on the nbd.

(Caution: same notation, but not the $g_{ij}(x)$ in vector bundle.)

Notation, most of the time we write $\langle \,,\, \rangle_x$ for g_x (or simply $\langle \,,\, \rangle$ for g)

Note: • By definition, $(g_{ij}(x))$ is a symmetric positive definite $n \times n$ matrix $\forall x \in U$.

• g can be regarded as a $(0,2)$-tensor satisfying, $g(X, X) \geq 0$, $\forall X \in \Gamma(TM)$

$$\begin{cases} g_x(X,X)=0 \iff X(x)=0 \\ g(X,Y)=g(Y,X),\ \forall X,Y\in\Gamma(TM) \end{cases}$$

Hence $\boxed{g=\sum_{i,j=1}^{n} g_{ij}(x)\,dx^i\otimes dx^j}$ in local coordinates

Def: A connection D (∇) on a C^∞ manifold M is a mapping

$$D: \Gamma(TM)\times\Gamma(TM) \longrightarrow \Gamma(TM)$$
$$(V, X) \longmapsto D_V X$$

such that

(C1) $D_{fV+gW} X = f D_V X + g D_W X$

(C2) $D_V(fX) = (Vf)X + f D_V X$

(C3) $D_V(X+Y) = D_V X + D_V Y$.

where $V, W, X, Y \in \Gamma(TM)$; $f, g\in C^\infty(M)$.

(and $Vf = D_V f$ is the directional derivative of f in the direction V.)

Note: $D_V X$ is called the covariant derivative of X in the direction of V. (or wrt V)

Fact: If $V, W \in \Gamma(TM)$ are vector fields such that

$V(x) = W(x)$, then $(D_V X)(x) = (D_W X)(x)$,
$\forall X \in \Gamma(TM)$.

(Pf: Ex!)

Using this fact, we have

Def: $\forall v \in T_x M$, one can define

$$D_v X \overset{\text{def}}{=\!=} (D_V X)(x) \quad (\in T_x M)$$

where V is a vector field s.t. $V(x) = v$.

eg: Standard connection on $\mathbb{R}^n$

Recall the directional derivative of function

$$D_v f = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

for smooth function defined near $x \in \mathbb{R}^n$.

A smooth vector field X on $\mathbb{R}^n$ can be written as

$$X = \sum X^i(x) \frac{\partial}{\partial x^i}$$

(x^i = standard coordinates on $\mathbb{R}^n$ $\Rightarrow \frac{\partial}{\partial x^i} = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ ith place)

where $X^i(x)$ are smooth functions.

Then $D_v X \overset{\text{def}}{=\!=} \sum D_v X^i(x) \frac{\partial}{\partial x^i}$ and

$$(D_V X)(x) \stackrel{def}{=} D_{V(x)} X$$

defines a connection on $\mathbb{R}^n$ (check: C1 – C3)

Clearly, for this standard connection on $\mathbb{R}^n$,

$$D_V\left(\frac{\partial}{\partial x^j}\right) = 0, \quad \forall j = 1, \cdots, n \text{ for the standard}$$

basis.

<u>Lemma</u>: The set of connections on M is <u>convex</u>.

i.e. If $D^1, \cdots, D^k$ are connections on M,

$f_1, \cdots, f_k$ are functions $\in C^\infty(M)$ with

$$\sum_{i=1}^k f_i = 1,$$

then $D = \sum_{i=1}^k f_i D^i$ is a connection on M.

$$\left(D_V X \stackrel{def}{=} \sum f_i D^i_V X \right)$$

<u>Pf</u>: C1 & C3 are clear (and do not need $\sum f_i = 1$)

For C2, we have

$$D_V(f X) = \sum_i f_i D^i_V(f X)$$

$$= \sum_i f_i \left[V(f) X + f D^i_V X \right]$$

$= V(f)X + f D_v(X)$ (by $\sum f_i = 1$)

$\#$

<u>Thm</u> Let M be a C^∞ manifold. Then $\exists$ a connection on M.

<u>Pf</u>: Let $\{(U_i, \phi_i)\}$ be an atlas on M.

Then $\{U_i\}$ is an open cover of M

$\Rightarrow \exists$ partitions of unity $\{\varphi_i\}$ subordinate to U_i

(WLOG, we may assume $\{V_k\}_{k\in\Lambda'} = \{U_i\}_{i\in\Lambda}$)

On each U_i, the standard connection on $\mathbb{R}^n$ induces a connection D^i. Then $\sum \varphi_i D^i$ is a connection on M by the previous lemma. $\#$

<u>Remark</u>: Similar argument shows that there exists Riemannian metric on any manifold.

<u>Lemma</u>: Let $v \in T_xM$ and $\gamma : [0, \varepsilon) \to M$ be a curve such that $\gamma'(0) = v$ (& $\gamma(0) = x$).

Suppose $X, Y \in \Gamma(TM)$ be two vector fields such that $X(\gamma(t)) = Y(\gamma(t)), \forall t$

Then $D_v X = D_v Y$.

(ie. $D_{\gamma'(0)} X$ is determined by $X \circ \gamma$.)

(Pf: Ex)

Thm: Let M = manifold

$g = \langle , \rangle$ = Riemannian metric on M

Then $\exists!$ connection D such that

(compatible with g) (L1) $X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$

(torsion free) (L2) $D_X Y - D_Y X - [X, Y] = 0$.

Pf: (Uniqueness)

In coordinates, any vector field can be written as

$$X = \sum X^i \frac{\partial}{\partial x^i}$$

$$\Rightarrow \quad D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad \text{for some } \Gamma_{ij}^k \text{ (functions)}$$

Now for $X = X^j \frac{\partial}{\partial x^j}$, $V = V^i \frac{\partial}{\partial x^i}$, then

$$D_V X = D_{V^i \frac{\partial}{\partial x^i}} \left(X^j \frac{\partial}{\partial x^j} \right)$$

$$= v^i D_{\frac{\partial}{\partial x^i}}\left(X^j \frac{\partial}{\partial x^j}\right)$$

$$= v^i \left[\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j} + X^j D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right]$$

$$= v^i \frac{\partial X^{\cancel{j}k}}{\partial x^i}\frac{\partial}{\partial x^{\cancel{j}k}} + v^i X^j\, \Gamma^k_{ij}\frac{\partial}{\partial x^k}$$

$$= v^i\left(\frac{\partial X^k}{\partial x^i} + X^j \Gamma^k_{ij}\right)\frac{\partial}{\partial x^k}$$

$\therefore \{\Gamma^k_{ij}\}$ determines $D_v X$.

Let $\quad g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle \quad \forall i, j$

$$\Rightarrow \quad \frac{\partial}{\partial x^i} g_{jk} = \frac{\partial}{\partial x^i}\left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right\rangle$$

$$= \left\langle D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right\rangle + \left\langle \frac{\partial}{\partial x^j}, D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right\rangle$$

$$= \left\langle \Gamma^l_{ij}\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right\rangle + \left\langle \frac{\partial}{\partial x^j}, \Gamma^l_{ik}\frac{\partial}{\partial x^l}\right\rangle$$

$$= \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl}$$

$$\Rightarrow \quad \left\{ \frac{\partial g_{jk}}{\partial x^i} = g_{lk}\Gamma^l_{ij} + g_{jl}\Gamma^l_{ik} \right. \qquad \text{——} \ (1)$$

$$\left\{\begin{aligned} \frac{\partial g_{ki}}{\partial x^j} &= g_{li}\Gamma^l_{jk} + g_{kl}\Gamma^l_{ji} \qquad \text{—— (2)} \\ \frac{\partial g_{ij}}{\partial x^k} &= g_{lj}\Gamma^l_{ki} + g_{il}\Gamma^l_{kj} \qquad \text{—— (3)} \end{aligned}\right.$$

By (L2)

$$0 = D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - D_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} - \cancelto{0}{\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]}$$

$$= \left(\Gamma^k_{ij} - \Gamma^k_{ji}\right)\frac{\partial}{\partial x^k}$$

$$\Rightarrow \qquad \Gamma^k_{ij} = \Gamma^k_{ji}, \qquad \forall\, i,j,k$$

Then $(1)+(2)-(3) \Rightarrow$

$$\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} = 2\, g_{kl}\Gamma^l_{ij}$$

Denote the inverse matrix of (g_{ij}) by (g^{ij}). Then

$$g^{sk}g_{kl} = \delta^s_l, \quad \forall\, s, l$$

$$\Rightarrow (\Gamma): \boxed{\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left[\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right]}$$

$\therefore \{\Gamma^k_{ij}\}$ & hence D satisfying L1 & L2 is uniquely determined by g.

(Existence): Let $\{(U_\beta, \phi_\beta)\}$ = atlas of M.

For $X = X^j \frac{\partial}{\partial x^j}$ & $V = V^i \frac{\partial}{\partial x^i}$ on U_β,

we define

$$D_V X \overset{\text{def}}{=\!=} V^i \left(\frac{\partial X^k}{\partial x^i} + \Gamma^k_{ij} X^j \right) \frac{\partial}{\partial x^k}$$

with Γ^k_{ij} defined by (Γ). Then one can check that $D_V X$ doesn't depend on the coordinate chart (U_β, ϕ_β). Hence it defines a connection D on M. The properties L1 & L2 are then easy to check. ※

<u>Note</u>:
- The connection given by this theorem is called the <u>Levi-Civita connection</u> of g (or <u>Riemannian connection</u> of g)
- The coefficients Γ^k_{ij} of D are called the

Christoffel symbols if D is Levi-Civita.

- The formula (Γ) is equivalent to

$$\langle D_X Y, Z\rangle = \frac{1}{2}\left\{ \begin{array}{l} X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ + \langle Z,[X,Y]\rangle + \langle Y,[Z,X]\rangle - \langle X,[Y,Z]\rangle \end{array} \right\}$$

for $X, Y, Z \in \Gamma(TM)$.

eg Fact: on S^3, there exist $\hat{i}, \hat{j}, \hat{k}$ orthonormal vector fields such that

$$[\hat{i},\hat{j}] = \hat{k}, \quad [\hat{j},\hat{k}] = \hat{i} \quad \& \quad [\hat{k},\hat{i}] = \hat{j}.$$

$$\langle D_{\hat{i}}\hat{j}, \hat{k}\rangle = \frac{1}{2}\left\{ \begin{array}{l} \cancelto{0}{\hat{i}\langle \hat{j},\hat{k}\rangle} + \cancelto{0}{\hat{j}\langle \hat{k},\hat{i}\rangle} - \cancelto{0}{\hat{k}\langle \hat{i},\hat{j}\rangle} \\ + \langle \hat{k},[\hat{i},\hat{j}]\rangle + \langle \hat{j},[\hat{k},\hat{i}]\rangle - \langle \hat{i},[\hat{j},\hat{k}]\rangle \end{array} \right\}$$

$$= \frac{1}{2}\left[\langle \hat{k},\hat{k}\rangle + \langle \hat{j},\hat{j}\rangle - \langle \hat{i},\hat{i}\rangle\right] = \frac{1}{2}.$$

Similarly $\quad \langle D_{\hat{i}}\hat{j}, \hat{i}\rangle = \langle D_{\hat{i}}\hat{j}, \hat{j}\rangle = 0$

Hence $\quad D_{\hat{i}}\hat{j} = \frac{1}{2}\hat{k}$

(Similarly for others (Ex!))

Geometry meaning of Levi-Civita connection

Def: Let N be a (embedded) submanifold of M. Suppose g is a metric on M, then the induce metric $\bar{g}$ of g on N is defined by

$$\bar{g}(X,Y) = g(X,Y), \quad \forall X,Y \in TN \subset TM$$

(eg. If $N \subset M$ is open, then $\bar{g} = g|_N$)

Def: Let (M,g) be a Riemannian manifold

D = Levi-Civita connection of g.

Suppose $N \subset M$ is a submanifold, then one can define a connection on N by

$$\bar{D}_X Y = (D_X Y)^{\perp}$$

where $(\ )_x^{\perp} : T_xM \to T_xN$ is the orthogonal projection (wrt g_x on T_xM).

Facts
- $\bar{D}$ is well-defined, ie $\bar{D}$ satisfies C1–C3
- $\bar{D}$ is the Levi-Civita connection of the

induce metric $\bar{g}$. (Pf: Ex!)

Note: If $M = \mathbb{R}^n$, g = standard metric (= flat metric) then Levi-Civita connection D = usual directional derivative on $\mathbb{R}^n$. Hence, the facts above give a geometric interpretation of the Levi-Civita connection on submanifolds N of $\mathbb{R}^n$.

"Meaning" of (L2): $D_X Y - D_Y X - [X, Y] = 0$.

(L2) doesn't use the metric g, and in local coordinates $\quad (L2) \Leftrightarrow \Gamma^k_{ij} = \Gamma^k_{ji}$

Hence, connections satisfying (L2) are called symmetric. Moreover, $T(X,Y) = D_X Y - D_Y X - [X, Y]$ defines a $(1,2)$-tensor on M called the torsion tensor, i.e. $T \in \Gamma(TM \otimes (\otimes^2 T^*M))$. Hence

$$\begin{aligned} D \text{ is symmetric } &\Leftrightarrow T \equiv 0 \\ &\Leftrightarrow D \text{ is torsion free.} \end{aligned}$$

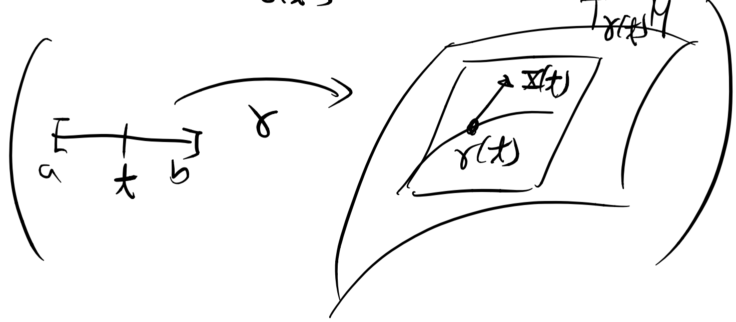
2.2 Parallel Transport

Let D be a connection (not necessary Levi-Civita) on M,

$\gamma : [a,b] \to M$ be an embedded curve such that $\gamma([a,b])$ is contained in a coordinate nbd. with coordinate functions $\{x^i\}$.

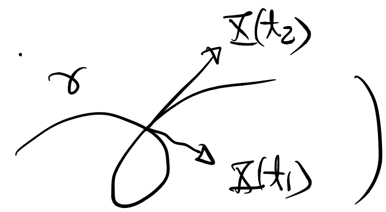
Suppose X is a vector field along γ, i.e. X depends smoothly on t and $X(t) \in T_{\gamma(t)}M$, $\forall t \in [a,b]$.

$$\begin{tikzcd} & TM \arrow[d] \\ {[a,b]} \arrow[ur, "X"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$



Since γ is embedded, X can be extended to a smooth vector field $\tilde{X}$ on M.

(Not true for immersed curve:



)

Now for any 2 extensions $\tilde{X}$ & $\tilde{Y}$, we must have

$$\tilde{X}(\gamma(t)) = \tilde{Y}(\gamma(t)) = X(t)$$

$$\Rightarrow \quad D_{\gamma'(t)} \tilde{X} = D_{\gamma'(t)} \tilde{Y}$$

$\therefore$ $\underline{D_{\gamma'(t)} X \text{ is well-defined}}$

In local coordinates,

$$\begin{cases} \gamma'(t) = \sum \gamma'^{i}(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} \\ X(t) = \sum X^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} \end{cases}$$

for some functions $\gamma'^{i}(t)$ & $X^i(t)$.

Then $$\boxed{D_{\gamma'(t)} X = \left(\frac{dX^k}{dt} + X^j \gamma'^{i} \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k}}$$

where Γ_{ij}^k are given by $D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}$.

Pf: $D_{\gamma'(t)} X = D_{\gamma'(t)} \left(X^j \frac{\partial}{\partial x^j} \right)$

$$= \left(D_{\gamma'(t)} X^j \right) \frac{\partial}{\partial x^j} + X^j D_{\gamma'(t)} \frac{\partial}{\partial x^j}$$

$$= \left(\frac{dX^k}{dt} + X^j \gamma'^{i} \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k}$$

#