Mathematical Analysis III

Tutorial 9 (November 21)

The following were discussed in the tutorial this week:

1. Recall the definition of total boundedness and separability.

2. A metric space is totally bounded if and only if every sequence has a Cauchy subsequence. In particular, a metric space is compact if and only if it is complete and totally bounded.

   Remark. *In the tutorial, what I wanted to say is "for all k, $(x_n^n)_{n=k}^\infty$ is a subsequence of $(x_n^k)_{n=1}^\infty$".*

3. If (X, d) is separable, then there are countably many open sets $(B_n)_{n\in\mathbb{N}}$ such that every open set $G \subset X$ is a union of some B_n's.

4. (Lindelöf Theorem) Let (X, d) be a separable metric space and $(G_\alpha)_{\alpha\in I}$ be a family of open sets. Then I has a countable subset $\bar{I}$ such that

$$\bigcup_{\alpha\in I} G_\alpha = \bigcup_{\alpha\in\bar{I}} G_\alpha.$$

5. Let c_0 be the space of all sequences of real numbers that converge to zero, that is

$$c_0 := \{(x_n)_{n=1}^\infty \in \ell^\infty : \lim_{n\to\infty} |x_n| = 0\}.$$

   Then c_0 is a separable subspace of $(\ell^\infty, \|\cdot\|_\infty)$.