MATH 3060 Mathematical Analysis III

Tutorial 2 (September 26)

The following were discussed in the tutorial this week:

1. **Convergence Criterion:** The Fourier series of a continuous, 2π-periodic function which is C^1-piecewise on $[-\pi, \pi]$ converges to the function uniformly.

   The definition of C^1-piecewise is recalled.

2. Let $f(x) = \cosh(x) := (e^x + e^{-x})/2$ on $[-\pi, \pi]$. Show that

$$f(x) \sim \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \cos nx \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{\pi \coth \pi + 1}{2}.$$

3. A simple application of Riemann-Lebesgue lemma gives the following slight generalization of the lemma:

   Let f be an integrable function on $[a, b]$. then

$$\lim_{n\to\infty} \int_a^b f(x) \cos nx dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_a^b f(x) \sin nx dx = 0.$$

4. By the generalization above, the exercise in tutorial 1 becomes:

   Let f be an integrable function on $[a, b]$. Show that

$$\lim_{n\to\infty} \int_a^b f(x) |\cos nx| dx = \frac{2}{\pi} \int_a^b f(x) dx.$$

5. In fact, we have the following more general result:

   Let f be an integrable function on $[a, b]$, and g be an integrable periodic function with period T. Then

$$\lim_{n\to\infty} \int_a^b f(x) g(nx) dx = \frac{1}{T} \int_0^T g(t) dt \int_a^b f(x) dx$$

   (**Hint:** First prove the result for constant function f.)

6. The following is left as an exercise:

   Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \in (0, 1]; \\ 0 & \text{if } x = 0. \end{cases}$$

   Show that f is Lipschitz continuous at any point in $[0, 1]$, but f does not satisfies the Lipschitz condition.