Solution 4

1. Draw the unit metric balls $B_r(0)$, $B_r^1(0)$ and $B_r^\infty(0)$ (with $r = 1$) for metrics d_2, d_1 and d_∞ on $\mathbb{R}^2$ respectively.

   Solution. The unit ball $B_1^2(0)$ is the standard one, the unit ball in d_∞-metric consists of points (x, y) either $|x|$ or $|y|$ is equal to 1 and $|x|, |y| \leq 1$, so $B_1^\infty(0)$ is the unit square. The unit ball $B_1^1(0)$ consists of points (x, y) satisfying $|x|+|y| \leq 1$, so the boundary is described by the curves $x + y = 1, x, y \geq 0$, $x - y = 1, x \geq 0, y \leq 0, -x + y = 1, x \leq 0, y \geq 0$, and $-x - y = 1, x, y \leq 0$. The result is the tilted square with vertices at $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$.

2. Let (X, d) be a metric space and define

$$\rho(x, y) \equiv \frac{d(x, y)}{1 + d(x, y)}.$$

   Show that

   (a) ρ is a metric on X.

   (b) A sequence converges in d if and only if it converges in ρ.

   (c) If ρ is equivalent to d, then $\exists\, C > 0$ such that $d(x, y) \leq C\ \forall x, y \in X$

   Solution.

   (a) M1 and M2 are obvious since d is a metric. To prove M3 consider the function $\phi(x) = x/(1 + x)$. We need to show that $a \leq b + c$ implies $\phi(a) \leq \phi(b) + \phi(c)$. First observe that ϕ is increasing so $\phi(a) \leq \phi(b + c)$ when $a \leq b + c$. Then

$$\begin{aligned} \phi(b+c) &= \frac{b+c}{1+b+c} \\ &= \frac{b}{1+b+c} + \frac{c}{1+b+c} \\ &\leq \frac{b}{1+b} + \frac{c}{1+c} \\ &= \phi(b) + \phi(c)\ , \end{aligned}$$

   (b) If $d(x_n, x) \to 0$, consider $0 \leq \rho(x_n, x) \leq d(x_n, x)$, result follows.
   If $\rho(x_n, y) \to 0$, then $d(x_n, x)$ is bounded by some $C > 0$. Consider

$$\frac{d(x_n, x)}{1 + C} \leq \frac{d(x_n, x)}{1 + d(x_n, x)} = \rho(x_n, x)$$

   Result follows.

   (c) If ρ is equivalent to d, then d is weaker than ρ. Hence, $\exists C > 0$ such that $d(x, y) \leq C\rho(x, y)\ \forall x, y \in X$. $\rho \leq 1$ obviously, result follows.

3. Give an example of two inequivalent metrics which have the same concept of convergence. i.e. convergence in $d \iff$ convergence in ρ.

   Solution. Consider d and ρ in the previous problem and take X be the real line and $d(x, y) = |x-y|$. Clearly d is stronger than ρ but they are not equivalent because $\rho(x, y) \leq 1, \forall x, y$. Yet it is clear that $x_n \to x$ in d if and only if it is so in ρ. It shows that two inequivalent metrics could induce the same topology on a set.

4. Show that d_2 is stronger than d_1 on $C[a,b]$ but they are not equivalent.

   Solution. Letting $f, g \in C[a,b]$, by Cauchy-Schwarz inequality,

$$d_1(f,g) = \int_a^b |f-g| \le \sqrt{\int_a^b 1}\sqrt{\int_a^b (f-g)^2} = \sqrt{b-a}\, d_2(f,g),$$

   so d_2 is stronger than d_1. Next, define f_n as an even function so that $f_n(x) = 0$ for $x \ge 1$, $f_n(0) = n^{3/4}$ and linear between $[0, 1/n]$. Then $\{f_n\}$ satisfies our requirement.

5. A "functional" is a real-valued function defined on a space of functions. Show that the following functionals are continuous with respect to the given metric. ($\mathbb{R}$ is always equipped with the standard metric $d(c,y) = |x-y|$)

   (a) $\Phi : (C[a,b], d_1) \to \mathbb{R}$ given by

$$\Phi(f) = \int_a^b \sqrt{1+f^2(x)}dx.$$

   (b) $\Phi : (C[a,b], d_\infty) \to \mathbb{R}$ with same Φ in (a).

   (c) $\Psi : (C[-1,1], d_\infty) \to \mathbb{R}$ given by

$$\Psi(f) = f(0).$$

   Solution.

   (a) Let $h(y) = \sqrt{1+y^2}$. Then $\Phi(f) = \int_a^b h(f)dx$. Since $h'(y) = \dfrac{y}{\sqrt{1+y^2}} \le 1$, one has, by the mean value theorem

$$\begin{aligned}|\Phi(f) - \Phi(g)| &\le \int_a^b |h(f) - h(g)|dx \le \int_a^b |f-g| \max_{s\in(g,f)} |h'(s)|dx \\ &\le \int_a^b |f-g|dx.\end{aligned}$$

   Hence it is continuous in $C[a,b]$ under the d_1-distance.

   (b) As d_∞ is stronger than d_1, the functional is also continuous in d_∞.

   (c) $|\Psi(f) - \Psi(g)| = |f(0) - g(0)| \le \max_{x\in[-1,1]} |f(x) - g(x)|$.
   Hence it is continuous in the d_∞-metric.

6. Show that $\Psi : (C[-1,1], d_1) \to \mathbb{R}$ given by $\Psi(f) = f(0)$ is not continuous.

   Solution. Let f_n be continuous function such that $f_n(x) = 1, x \in [-1/n, 1/n]; f_n(x) = 0, x \in [-2/n, 2/n]$, and $0 \le f_n \le 1$. Then $\Psi(f_n) = 1$ but $f_n \to 0$ in the d_1-metric.