Applications of Baire Category Theorem

Application (1)

Prop 4.16 The set of all continuous, nowhere differentiable functions forms a set of 2^{nd} category in $C[a,b]$, and hence dense in $C[a,b]$.

To prove this, we need the following lemma:

Lemma 4.15 Let $f \in C[a,b]$ be differentiable at $x \in [a,b]$. Then it is Lipschitz continuous at x.

Pf: By differentiability at x, for $\varepsilon = 1$, $\exists\, \delta_0 > 0$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < 1, \quad \forall\, y \text{ s.t. } 0 < |y - x| < \delta_0$$

$$\Rightarrow \quad |f(y) - f(x)| < (1 + |f'(x)|)\, |y - x| \quad \forall\, y \text{ s.t. } |y - x| < \delta_0.$$

For $|y - x| \geq \delta_0$, we have

$$|f(y)-f(x)| \leq |f(y)|+|f(x)|$$
$$\leq 2\|f\|_\infty \leq \left(\frac{2\|f\|_\infty}{\delta_0}\right)|y-x|$$

$\therefore$ If $L = \max\left\{ 1+|f'(x)|, \frac{2\|f\|_\infty}{\delta_0} \right\}$, then

$$|f(y)-f(x)| \leq L|y-x|, \quad \forall y \in [a,b]. \#$$

<u>Pf of Prop 4.16</u>:

For each $L>0$, define

$$S_L = \left\{ f \in C[a,b] : \begin{array}{l} f \text{ Lip. cts. at } \underline{\text{some}}\ x \\ \text{with Lip. constant} \leq L \end{array} \right\}$$

<u>Claim (1)</u> S_L is a closed set.

<u>Pf</u>: Let $\{f_n\}$ be a seq. in S_L converges uniformly to $f \in C[a,b]$.

$f_n \in S_L \Rightarrow \exists\, x_n \in [a,b]$ s.t.

f_n is Lip. cts at x_n with Lip constant $\leq L$.

Passing to a subseq., we may assume

$$x_n \to x^* \in [a,b].$$

Then $|f(y)-f(x^*)|$

$$\le |f(y)-f_n(y)| + |f_n(y)-f_n(x_n)|$$

$$+|f_n(x_n)-f(x^*)| + |f_n(x^*)-f(x^*)|$$

$$\le |f(y)-f_n(y)| + |f_n(x^*)-f(x^*)|$$

$$+ L|y-x_n| + L|x_n-x^*|$$

$$\to L|y-x^*| \quad \text{as } n\to\infty.$$

$\therefore$ $f \in S_L$. Hence S_L is closed. $\#$

Claim(2): S_L is nowhere dense.

Pf: Let $f \in S_L$, we need to show that $\forall \varepsilon>0$, $\exists g \in C[a,b]$ such that

$$\|f-g\|_\infty < \varepsilon \text{ and } g \notin S_L.$$

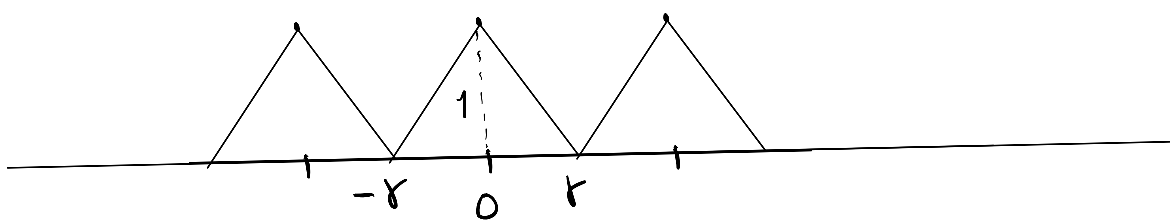
By Weierstrass approximation theorem, $\exists$ a polynomial $p(x)$ such that

$$\|f-p\|_\infty < \varepsilon/2.$$

Then define $g(x) = p(x) + \frac{\varepsilon}{2}\varphi_r(x)$,

where $\varphi_r : \mathbb{R} \to [0,1]$ with $\varphi_r(0)=1$

defined by :



Then $\|f-g\|_\infty \le \|f-p\|_\infty + \|\frac{\varepsilon}{2}\varphi_r(x)\|_\infty$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (\text{for any } r>0)$$

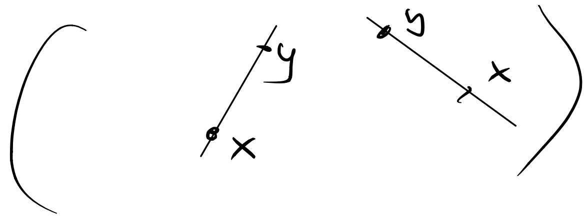
On the other hand, since polynomail $p(x)$ is Lip cts. Let its Lip const. be $L_1 > 0$.

Then $|g(y)-g(x)| \ge \frac{\varepsilon}{2}|\varphi_r(y)-\varphi_r(x)| - |p(y)-p(x)|$

$$\ge \frac{\varepsilon}{2}|\varphi_r(y)-\varphi_r(x)| - L_1|y-x|$$

Hence $\forall x \in [a,b]$, $\exists y \in [a,b]$ near x such that

$$|g(y)-g(x)| \geq \frac{\varepsilon}{2}\frac{1}{r}|y-x| - L_1|y-x|$$



$$= \left(\frac{\varepsilon}{2r} - L_1\right)|y-x|.$$

Therefore, if $r>0$ small enough s.t. $\frac{\varepsilon}{2r} - L_1 > L$, then $\forall\, x \in [a,b]$, $\exists\, y \in [a,b]$ such that

$$|g(y)-g(x)| > L|y-x|.$$

Hence $g \notin S_L$ #

Claims (1) & (2) $\Rightarrow$ $\bigcup_{k=1}^{\infty} S_k$ is of 1st category.

By Lemma 4.15, the set

$$S = \{f \in C[a,b] : \text{differentiable at some } x \in [a,b]\}$$

is contained in $\bigcup_{k=1}^{\infty} S_k$.

$\Rightarrow$ S is of 1st category

$\Rightarrow$ $C[a,b] \setminus S$ is of 2^{nd} category

and hence dense in $C[a,b]$. ※

Application (2)

Prop 4.17 Any basis of an infinite dimensional Banach space consists of uncountablly many vectors.

(complete normed space)

Pf: Suppose on the contrary that $\mathcal{B} = \{w_i\}$ is countable basis of an infinite dimensional Banach space V.

Let $W_n = \text{span}\{w_1, \cdots, w_n\}$

Then $V = \bigcup_{n=1}^{\infty} W_n$.

Claim (1) W_n is closed.

(sketch of Pf): $W_n = \text{span}\{w_1, \cdots, w_n\}$ is of dimension n $\cong \mathbb{R}^n$ as vector space.

As any 2 norms in $\mathbb{R}^n$ are equivalent and $\mathbb{R}^n$ is complete, W_n is closed. #

<u>Claim (2)</u> W_n is nowhere dense.

Pf: $\dim V = +\infty \Rightarrow \exists\, v_0 \notin W_n$ with $\|v_0\| = 1$.

Then $\forall\, w \in W_n$,

$$w + \frac{\varepsilon}{2} v_0 \in B_\varepsilon(w) \cap (V \setminus W_n)$$

$\therefore B_\varepsilon(w) \cap (V \setminus W_n) \neq \phi$ #

Therefore, $V = \bigcup_{n=1}^{\infty} W_n$ contradicts the Baire Category Thm. $\therefore \mathcal{B}$ cannot form a basis. #