Prop 2.1 Let $f:(X,d)\to(Y,\rho)$ be a mapping between 2 metric spaces, and $x_0\in X$. Then f is continuous at x_0 $\iff$ $\forall \varepsilon>0, \exists \delta>0$ such that

$$\rho(f(x),f(x_0))<\varepsilon, \ \forall x \text{ with } d(x,x_0)<\delta.$$

(Pf: Ex!)

Prop 2.2: Let $f:(X,d)\to(Y,\rho)$ &

$$g:(Y,\rho)\to(Z,m)$$

are mappings between metric spaces.

(a) If f is continuous at x & g is continuous at $f(x)$, then $g\circ f:(X,d)\to(Z,m)$ is continuous at x.

(b) If f is cts in X and g is cts in Y, then $g\circ f$ is cts in X.

(Pf: Easy)

Eg: Let (X,d) be a metric space, $A\subset X$, $A\neq\emptyset$.

Define $\rho_A: X\to\mathbb{R}$ by

$$\rho_A(x)=\inf_{y\in A} d(y,x)$$

(distance from x to the subset A).

<u>Claim:</u> $|\rho_A(x) - \rho_A(y)| \leq d(x,y)$, $\forall x, y \in X$.

<u>Pf of claim:</u> For fixed $x, y \in X$.

By defn. of $\rho_A(y)$,

$$\forall \varepsilon > 0, \ \exists z \in A \text{ s.t. } \rho_A(y) + \varepsilon > d(z,y)$$

Hence, $\rho_A(x) \leq d(z,x) \leq d(x,y) + d(y,z)$
$< d(x,y) + \rho_A(y) + \varepsilon$

$$\Rightarrow \quad \rho_A(x) - \rho_A(y) < d(x,y) + \varepsilon.$$

Interchanging the roles of x & y

$$\rho_A(y) - \rho_A(x) < d(x,y) + \varepsilon$$

Therefore $|\rho_A(x) - \rho_A(y)| < d(x,y) + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $|\rho_A(x) - \rho_A(y)| \leq d(x,y)$

\#

By claim, $d(x_n, x) \to 0 \Rightarrow \rho_A(x_n) \to \rho_A(x)$

$\therefore \quad \rho_A : (X, d) \to \mathbb{R}$ is cts.

(In fact, ρ_A is "Lipschitz continuous".)

This example shows that there are "many" cts functions on a metric space.

<u>Notation:</u> Usually, we use the following notations

$$d(x,F) = \inf\{d(x,y) : y \in F\}$$

$$d(E,F) = \inf\{d(x,y) : x \in E, y \in F\}$$

for subsets E & F.

---

§2.2 Open and Closed Sets

Def: Let (X,d) = metric space

- A set $G \subset X$ is called an open set if $\forall\, x \in G$, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) = \{y : d(x,y) < \varepsilon\} \subset G$.

  (The number $\varepsilon > 0$ may vary depending on x.)

- We also define the empty set $\emptyset$ to be an open set.

Prop: Let (X,d) be a metric space. We have

(a) X and $\emptyset$ are open sets.

(b) Arbitrary union of open sets is open; if G_α, $\alpha \in A$, is a collection of open sets then $\bigcup_{\alpha \in A} G_\alpha$ is an open set.

(c) Finite intersection of open sets is open:

if $G_1, \cdots, G_N$ are open sets, then $\bigcap_{j=1}^{N} G_j$ is an open set.

<u>Pf</u>: (a) Clear

(b) Let $x \in \bigcup_{\alpha \in \mathcal{A}} G_\alpha$

$\Rightarrow x \in G_\alpha$ for some $\alpha \in \mathcal{A}$.

$\Rightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset G_\alpha$ (since G_α open)

$\Rightarrow B_\varepsilon(x) \subset \bigcup_{\alpha \in \mathcal{A}} G_\alpha$. ※

(c) Let $x \in \bigcap_{j=1}^{N} G_j$

$\Rightarrow x \in G_j, \ \forall j = 1, \cdots, N$

$\Rightarrow \exists \varepsilon_j > 0$ s.t. $B_{\varepsilon_j}(x) \subset G_j, \ \forall j = 1, \cdots, N$,

Let $\varepsilon = \min\{\varepsilon_1, \cdots, \varepsilon_N\} > 0$. Then

$$B_\varepsilon(x) \subset B_{\varepsilon_j}(x) \subset G_j, \ \forall j = 1, \cdots, N$$

$$\Rightarrow B_\varepsilon(x) \subset \bigcap_{j=1}^{N} G_j \quad ※$$

<u>Def</u>: Let (X, d) be a metric space.

A set $F \subset X$ is called a <u>closed set</u> if the complement $X \setminus F$ is an open set.

<u>Prop 2.4</u> Let (X, d) be a metric space. We have

(a) X and $\emptyset$ are closed sets.

(b) Arbitrary intersection of closed sets is closed:

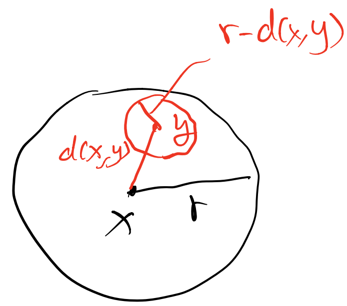
if F_α, $\alpha \in \mathcal{A}$, are closed sets, then $\bigcap_{\alpha \in \mathcal{A}} F_\alpha$ is closed.

(c) Finite union of closed sets is closed:

if $F_1, \cdots, F_N$ are closed sets, then $\bigcup_{j=1}^{N} F_j$ is closed.

<u>Note:</u> Prop 2.3 & 2.4 $\Rightarrow$ X & $\emptyset$ are both open & closed.

<u>eg 2.8</u> (1) Every metric ball $B_r(x) = \{ y \in X : d(y,x) < r \}$ is an open set.



<u>Pf</u>: $\forall\, y \in B_r(x)$

Then $\varepsilon = r - d(x,y) > 0$

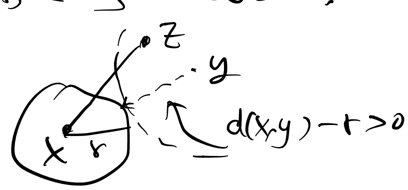
& $\forall\, z \in B_\varepsilon(y)$,

$$d(z,x) \le d(z,y) + d(y,x) < \varepsilon + d(y,x) = r$$

$\Rightarrow$ $B_\varepsilon(y) \subset B_r(x)$ ※

(2) The set $E = \{ y \in X : d(y,x) > r \}$ is open and hence

$X \setminus E = \{ y \in X : d(y,x) \le r \}$ is closed.



<u>Pf</u>: $\forall\, y \in E$

Then $\varepsilon = d(x,y) - r > 0$

$\forall z \in B_\varepsilon(y)$,

$$d(z,x) \geq d(x,y) - d(z,y) \quad \text{(triangle ineq.)}$$
$$> d(x,y) - (d(x,y) - r)$$
$$= r$$

$\therefore \quad B_\varepsilon(y) \subset E$ #

Note: We usually write (Confusing notation here, not equal closure of $B_r(x)$ in general)

$\overline{B_r}(x) = \overline{B_r(x)} = \{y \in X : d(y,x) \leq r\}$

the closed ball of radius r centered at x.

(3) Since $B_r(x)$ & $E = \{y \in X : d(x,y) > r\}$ are open,

$B_r(x) \cup E$ is open.

$\Rightarrow \quad X \setminus (B_r(x) \cup E) = \{y \in X : d(x,y) = r\}$

is closed.

In particular, $E = \{y \in X : d(y,x) > 0\}$ is open

$\Rightarrow \quad \{x\} = X \setminus E$ is closed (in any metric space)

[Note: $\{x\}$ may not open (unless $\exists\, \varepsilon_0 > 0$ s.t. $B_{\varepsilon_0}(x) = \{x\}$.)]

eg 2.9 $B_{\frac{1}{n}}(x)$, $n = 1, 2, \cdots$, are open sets.

Claim: $\bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$ (closed, may not open)

($\therefore$ infinite intersection of open sets may not open.)

Pf of claim : $\forall y \in \bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) \Rightarrow y \in B_{\frac{1}{n}}(x)$, $\forall n=1,2,\cdots$

$$\Rightarrow d(y,x) < \frac{1}{n}, \quad \forall n=1,2,\cdots$$

$$\Rightarrow d(y,x) = 0.$$

$$\Rightarrow y = x \quad ✻$$

eg 2.11 $X = C[a,b]$ with $d_\infty(f,g) = \|f-g\|_\infty = \sup_{x\in[a,b]} |f-g|(x)$

let $E = \{f \in C[a,b] : f(x) > 0, \forall x \in [a,b]\} \subset X$

$\forall f \in E$, f is positive, cts on the closed & bounded interval $[a,b]$, therefore $\exists\, m > 0$ s.t.

$$f(x) \geq m > 0, \quad \forall x \in [a,b].$$

Consider $B^\infty_{\frac{m}{2}}(f) = \{g \in C[a,b] : d_\infty(g,f) < \frac{m}{2}\}$.

$\forall g \in B^\infty_{\frac{m}{2}}(f)$, we have $\forall x \in [a,b]$

$$g(x) = [g(x) - f(x)] + f(x)$$

$$\geq f(x) - \|g-f\|_\infty$$

$$> f(x) - \frac{m}{2} \geq m - \frac{m}{2} = \frac{m}{2} > 0$$

$\therefore g \in E$ & hence $B^\infty_{\frac{m}{2}}(f) \subset E$.

$\therefore E$ is open in $(C[a,b], d_\infty)$.

Similarly, one can show that $\forall \alpha \in \mathbb{R}$

$$\{f \in C[a,b] : f(x) > \alpha, \ \forall x \in [a,b]\}$$
$$\{f \in C[a,b] : f(x) < \alpha, \ \forall x \in [a,b]\}$$

are open in $(C[a,b], d_\infty)$.

And
$$\{f \in C[a,b] : f(x) \geq \alpha, \ \forall x \in [a,b]\}$$
$$\{f \in C[a,b] : f(x) \leq \alpha, \ \forall x \in [a,b]\}$$

are closed in $(C[a,b], d_\infty)$

eg 2.12: Let $X \neq \emptyset$ and d = discrete metric on X.

Then $\forall$ subset $E \subset X$,

$$B_{\frac{1}{2}}(x) = \{x\} \subset E, \ \forall x \in E.$$

$\therefore$ E is open.

Therefore, any subset E of $(X, \text{discrete})$ is open,

& hence any subset E of $(X, \text{discrete})$ is closed.

Together, any subset E of $(X, \text{discrete})$ is both open and closed.

In particular, any $\{x\} \subset (X, \text{discrete})$ is both open and closed.

Prop 2.5 Let (X, d) be a metric space. A sequence $\{x_n\}$ converges to x <u>if and only if</u>

$\forall$ open set G containing x, $\exists\, n_0$ such that

$$x_n \in G, \ \forall\, n \geq n_0 .$$

Pf: $(\Rightarrow)$ Let G open & $x \in G$

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset G$

As $x_n \to x$, for this $\varepsilon > 0$, $\exists\, n_0$ s.t.

$$d(x_n, x) < \varepsilon, \ \forall\, n \geq n_0$$

$$\Rightarrow \quad x_n \in B_\varepsilon(x) \subset G, \quad \forall\, n \geq n_0$$

$(\Leftarrow)$ $\forall\, \varepsilon > 0$, $B_\varepsilon(x)$ is an open set containing x.

Therefore $\exists\, n_0$ s.t. $x_n \in B_\varepsilon(x)$, $\forall\, n \geq n_0$

$$\Rightarrow \quad d(x_n, x) < \varepsilon, \ \forall\, n \geq n_0$$

$\#$

Prop 2.6 Let (X, d) be a metric space. Then a set $A \subset X$ is closed <u>if and only if</u> whenever $\{x_n\} \subset A$ and $x_n \to x$ as $n \to \infty$ implies that $x \in A$.

Pf: $(\Rightarrow)$ Suppose <u>not</u>. Then $x \notin A$

i.e. $x \in X \setminus A$ which is open (as A closed)

$\Rightarrow \exists \varepsilon > 0,\ B_\varepsilon(x) \subset X \setminus A$.

On the other hand $x_n \to x$, $\exists n_0$ s.t. $d(x_n, x) < \varepsilon$ $\forall n \geq n_0$

$\Rightarrow x_n \in B_\varepsilon(x) \subset X \setminus A$

$\Rightarrow x_n \notin A$ contradiction $\#$

$(\Leftarrow)$ Suppose <u>not</u>. Then A is not closed.

$\Leftrightarrow X \setminus A$ is not open

$\exists x \in X \setminus A$ s.t. $B_\varepsilon(x) \not\subset X \setminus A$, $\forall \varepsilon > 0$.

In particular, $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$, $\forall n = 1, 2, \cdots$

Pick $x_n \in B_{\frac{1}{n}}(x) \cap A$ for each n

Then $\{x_n\} \subset A$ & $d(x_n, x) < \frac{1}{n}$, $\forall n$

$\Rightarrow x_n \to x$ as $n \to \infty$.

Contracting the assumption (as $x \in X \setminus A$)

$\#$